

Multi-portfolio time consistency for set-valued convex and coherent risk measures

Zachary Feinstein ^{*} Birgit Rudloff [†]

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Abstract

Equivalent characterizations of multi-portfolio time consistency are deduced for closed convex and coherent set-valued risk measures in $L_d^p(\mathcal{F}_T)$ with $1 \leq p \leq \infty$ and image space in the power set of $L_d^p(\mathcal{F}_t)$. In the convex case, multi-portfolio time consistency is equivalent to a condition on the sum of minimal penalty functions. In the coherent case, multi-portfolio time consistency is equivalent to a generalized version of stability of the dual variables. As examples, the set of superhedging portfolios in markets with transaction costs is shown to have the stability property and a multi-portfolio time consistent version of the set-valued average value at risk, the composed AV@R, is given and its dual representation deduced.

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1 Introduction

The use of risk measures to calculate capital requirements has been widely studied, beginning with the seminal work on coherent risk measures by Artzner et al. [2, 3]. In [18, 20] the axioms of coherency have been relaxed to define convex risk measures.

Dynamic risk measures arise in a multi-period setting where risk is defined conditionally on information known at time t described by a filtration $(\mathcal{F}_t)_{t=0}^T$. Time consistency is a useful property for dynamic risk measures; it gives a relation between risks at different times. Conceptually, a risk measure is time consistent if, a priori, it is known that at a future time one portfolio is more risky than another then at any prior time the same relation holds as well. For dynamic risk measures and time consistency in the scalar setting, we refer to [4, 34, 15, 10, 35, 7, 17, 12, 11, 1, 19] for the discrete time case and [21, 13, 14] for the continuous time case. In particular, an equivalent property for time consistency in the coherent case is given by the stability of the dual probability measures as seen in [1, 10, 19, 17, 4]. In the convex case a property on the sum of penalty functions was deduced in [17, 10, 8, 9, 1]. This property is referred to as the cocycle property in [8, 9].

^{*}Princeton University, Department of Operations Research and Financial Engineering, Princeton, NJ 08544, USA. Supported by NSF RTG grant 0739195.

[†]Princeton University, Department of Operations Research and Financial Engineering; and Bendheim Center for Finance, Princeton, NJ 08544, USA, brudloff@princeton.edu. Research supported by NSF award DMS-1007938.

When multivariate random variables, or markets with transaction costs, are considered it becomes natural to work with set-valued risk measures; in this way capital requirements can be made in a basket of currencies or assets rather than a chosen numéraire. In the static single period framework set-valued risk measures have been studied in [28, 25, 23, 24]. The dynamic version of set-valued risk measures were studied in [16, 5], and an approach to dynamic risk measures under transaction costs using a family of scalar functions was studied in [6, 27]. We will take our setting from [16]. In that paper, set-valued coherent and convex dynamic risk measures were discussed, and a set-valued version of time consistency, called multi-portfolio time consistency, was defined. In the scalar framework, time consistency is equivalent to the recursive form $\rho_t(X) = \rho_t(-\rho_{t+1}(X))$, and in the set-valued framework it was proven that multi-portfolio time consistency is equivalent to the set-valued recursive form $R_t(X) = R_t(-R_{t+1}(X)) := \bigcup_{Z \in R_{t+1}(X)} R_t(-Z)$.

In section 2, we will review the basic results of [16] that are needed for the present paper. Section 3 deduces an equivalent characterization of multi-portfolio time consistency for set-valued normalized closed convex risk measures. This is given by a property on the sum of minimal penalty functions, called the cocycle property in the scalar case in [8, 9], and is the extension of the scalar result of [17, 10, 8, 9, 1]. The proof of this results is entirely different from the proof in the scalar case as the scalar method cannot be applied in the set-valued case. Section 4 discusses two equivalent characterizations of multi-portfolio time consistency for set-valued normalized closed coherent risk measures. The first is the result for convex risk measures applied to the coherent case. This characterization has not been explicitly stated in the scalar case, but is useful for generating multi-portfolio time consistent risk measures (see e.g. [11]). The second property is the set-valued generalization of stability of the dual variables, and generalizes the work in [1, 10, 19, 17, 4]. Section 5 gives a method for composing a risk measure backwards in time to create a multi-portfolio time consistent version of this risk measure. Special attention is given to the composed form for closed convex and coherent risk measures. Finally, in section 6, we discuss two examples. First, we look at the set of superhedging portfolios and show that the set of dual variables for this closed coherent risk measure is stable in the sense given in section 4. Then we study the dynamic set-valued average value at risk, proposed in [26, 16], and give the multi-portfolio time consistent version of it by backward composition. We use the results of section 5 to deduce its dual representation.

2 Set-valued dynamic risk measures

In this section, we will introduce some notations and, for easing the readability of the present paper, review basic definitions and main results about duality and multi-portfolio time consistency of set-valued dynamic risk measures from [16].

Consider a filtered probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$ satisfying the usual conditions where \mathcal{F}_0 is the trivial sigma algebra. Let $|\cdot|$ denote an arbitrary norm in \mathbb{R}^d and let $L_d^p(\mathcal{F}_t) := L_d^p(\Omega, \mathcal{F}_t, \mathbb{P})$ for any $p \in [1, \infty]$. $L_d^p(\mathcal{F}_t)$ denotes the linear space of the equivalence classes of \mathcal{F}_t -measurable functions $X : \Omega \rightarrow \mathbb{R}^d$ such that $\|X\|_p = (\int_{\Omega} |X(\omega)|^p d\mathbb{P})^{\frac{1}{p}} < +\infty$ for $p \in [1, \infty)$, and $\|X\|_{\infty} = \text{ess sup}_{\omega \in \Omega} |X(\omega)| < +\infty$ for $p = \infty$. We consider the dual pair $(L_d^p(\mathcal{F}_t), L_d^q(\mathcal{F}_T))$ with $p \in [1, +\infty]$ and q is such that $\frac{1}{p} + \frac{1}{q} = 1$, and endow it with the norm topology, respectively the $\sigma(L_d^{\infty}(\mathcal{F}_T), L_d^1(\mathcal{F}_T))$ -topology on $L_d^{\infty}(\mathcal{F}_T)$ in the case $p = +\infty$.

We denote by $L_d^p(\mathcal{F}_t; D_t)$ those random vectors in $L_d^p(\mathcal{F}_t)$ that take \mathbb{P} -a.s. values in D_t . Note that an element $X \in L_d^p(\mathcal{F}_t)$ has components X_1, \dots, X_d in $L^p(\mathcal{F}_t) = L_1^p(\mathcal{F}_t)$. (In-)equalities between random vectors are always understood componentwise in the \mathbb{P} -a.s. sense. The multiplication between a random variable $\lambda \in L^{\infty}(\mathcal{F}_t)$ and a set of random vectors $D \subseteq L_d^p(\mathcal{F}_T)$ is

understood as $\lambda D = \{\lambda Y : Y \in D\} \subseteq L_d^p(\mathcal{F}_T)$ with $(\lambda Y)(\omega) = \lambda(\omega)Y(\omega)$.

Let $L_d^p(\mathcal{F}_t)_+ := \{X \in L_d^p(\mathcal{F}_t) : X \in \mathbb{R}_+^d \text{ } \mathbb{P} - \text{a.s.}\}$ denote the closed convex cone of \mathbb{R}^d -valued \mathcal{F}_t -measurable random vectors with \mathbb{P} -a.s. non-negative components. Additionally we write $L_d^p(\mathcal{F}_t)_{++} := \{X \in L_d^p(\mathcal{F}_t) : X \in \mathbb{R}_{++}^d \text{ } \mathbb{P} - \text{a.s.}\}$ to be the \mathcal{F}_t -measurable random vectors which are \mathbb{P} -a.s. positive.

As in [29] and discussed in [36, 30], the portfolios in this paper are in “physical units” of an asset rather than the value in a fixed numéraire via some price. That is, for a portfolio $X \in L_d^p(\mathcal{F}_t)$, the values of X_i (for $1 \leq i \leq d$) are the number of units of asset i in the portfolio at time t .

Let \tilde{M}_t be an \mathcal{F}_t -measurable set, where $\tilde{M}_t[\omega]$ denotes the set of eligible portfolios, i.e. those portfolios which can be used to compensate for the risk of a portfolio, at time t and state ω . We assume $\tilde{M}_t[\omega]$ is a subspace of \mathbb{R}^d for almost every $\omega \in \Omega$. It then follows that $M_t := L_d^p(\mathcal{F}_t; \tilde{M}_t)$ is a closed (weak* closed if $p = +\infty$) linear subspace of $L_d^p(\mathcal{F}_t)$, see section 5.4 and proposition 5.5.1 in [30]. As noted in [16], it will commonly be the case that the same portfolios are eligible through time and independent of the state of the market, that is $\tilde{M}_t = M_0$ almost surely for all times t for some subspace M_0 of \mathbb{R}^d . We will assume that this is the case for most of the results in the present paper. Notice that this assumption also implies $M_t \subseteq M_{t+1}$ for all times t , which is an assumption appearing in the equivalent characterizations of multi-portfolio time consistency in [16]. Let us denote $(M_t)_+ := M_t \cap L_d^p(\mathcal{F}_t)_+$. We will assume $(M_t)_+$ is nontrivial, i.e. $(M_t)_+ \neq \{0\}$.

A conditional risk measure is a function which maps a d -dimensional random variable X into

$$\mathcal{P}(M_t; (M_t)_+) := \{D \subseteq M_t : D = D + (M_t)_+\},$$

which is a subset of the power set 2^{M_t} . Conceptually, the value of a risk measure $R_t(X)$ is the collection of eligible portfolios at time t which cover the risk of the portfolio X .

Definition 2.1. A function $R_t : L_d^p(\mathcal{F}_T) \rightarrow \mathcal{P}(M_t; (M_t)_+)$ is a **normalized conditional risk measure** at time t if it is

1. M_t -translative: $\forall m_t \in M_t : R_t(X + m_t) = R_t(X) - m_t$;
2. $L_d^p(\mathcal{F}_T)_+$ -monotone: $Y - X \in L_d^p(\mathcal{F}_T)_+ \Rightarrow R_t(Y) \supseteq R_t(X)$;
3. finite at zero: $\emptyset \neq R_t(0) \neq M_t$;
4. normalized: $\forall X \in L_d^p(\mathcal{F}_t) : R_t(X) = R_t(X) + R_t(0)$.

Additionally, a conditional risk measure at time t is **convex** if for all $X, Y \in L_d^p(\mathcal{F}_T)$, for all $0 \leq \lambda \leq 1$

$$R_t(\lambda X + (1 - \lambda)Y) \supseteq \lambda R_t(X) + (1 - \lambda)R_t(Y),$$

is **positive homogeneous** if for all $X \in L_d^p(\mathcal{F}_T)$, for all $\lambda > 0$

$$R_t(\lambda X) = \lambda R_t(X),$$

and is **coherent** if it is convex and positive homogeneous.

A conditional risk measure at time t is **closed** if the graph of the risk measure,

$$\text{graph } R_t = \{(X, u) \in L_d^p(\mathcal{F}_T) \times M_t : u \in R_t(X)\},$$

is closed in the product topology.

The properties given in definition 2.1 above are desirable for risk measures (as discussed in [24, 16]). M_t -translativity gives the interpretation of a risk measure as the ‘capital requirement’ to cover risk. Monotonicity means that if one portfolio dominates another (almost surely) then its

risk should be lower. The normalization property (with closedness) ensures that the zero portfolio compensates the risk of the zero payoff. Additionally, convexity (and coherence) are useful properties for measuring diversification effects of portfolios. Similarly, one can define conditional convexity, where $\lambda \in \mathbb{R}$ above is replaced by $\lambda \in L^\infty(\mathcal{F}_t)$ s.t. $0 \leq \lambda \leq 1$, and conditional positive homogeneity, where $\lambda > 0$ above is replaced by $\lambda \in L^\infty(\mathcal{F}_t)_{++}$. Clearly, a conditional convex (conditional positive homogeneous) function is also convex (positive homogeneous). We are working in this paper with the more general concept as all the results deduced in this paper hold true in the more general setting as well.

The image space of a closed convex conditional risk measure is given by

$$\mathcal{G}(M_t; (M_t)_+) = \{D \subseteq M_t : D = \text{cl co}(D + (M_t)_+)\}.$$

A **dynamic risk measure** $(R_t)_{t=0}^T$ is a sequence of conditional risk measures. It is said to have one of the properties given in definition 2.1 if R_t has this property for every $t \in \{0, 1, \dots, T\}$.

Instead of considering risk measures directly, a portfolio manager might be interested in the set of portfolios which have an “acceptable” level of risk, called an acceptance set.

Definition 2.2. $A_t \subseteq L_d^p(\mathcal{F}_T)$ is a **conditional acceptance set** at time t if it satisfies all the following:

1. $M_t \cap A_t \neq \emptyset$,
2. $M_t \cap (L_d^p(\mathcal{F}_T) \setminus A_t) \neq \emptyset$, and
3. $A_t + L_d^p(\mathcal{F}_T)_+ \subseteq A_t$.

A conditional acceptance set is **normalized** if it satisfies $A_t + A_t \cap M_t \subseteq A_t$ with $0 \in \text{cl}(A_t \cap M_t)$.

There is a one-to-one correspondence between risk measures and acceptance sets, see remark 2 and proposition 2.11 in [16]. The acceptance set associated with a conditional risk measure R_t is defined by

$$A_t := \{X \in L_d^p(\mathcal{F}_T) : 0 \in R_t(X)\}$$

and the risk measure associated with an acceptance set is defined by

$$R_t(X) := \{u \in M_t : X + u \in A_t\}.$$

Further, we will define the s -stepped acceptance set at time t as

$$A_{t,t+s}^{M_{t+s}} := \{X \in M_{t+s} : 0 \in R_t(X)\} = A_t \cap M_{t+s}.$$

For a thorough discussion of stepped risk measures see section 8.3.

2.1 Dual representation

Let $\mathcal{M}_d(\mathbb{P})$ denote the set of d -dimensional probability measures absolutely continuous with respect to \mathbb{P} . Let $\text{diag}(x)$ be the diagonal matrix with the components of x on the main diagonal. Consider $\mathbb{Q} \in \mathcal{M}_d(\mathbb{P})$. We will use a \mathbb{P} -almost sure version of the \mathbb{Q} -conditional expectation given by

$$\mathbb{E}^{\mathbb{Q}}[X | \mathcal{F}_t] := \mathbb{E} \left[\text{diag} \left(\mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_t \right] \right)^{-1} \text{diag} \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) X \middle| \mathcal{F}_t \right],$$

where $\mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_t \right]^{-1}(\omega) \frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) := 1$ for any $\omega \in \Omega$ with $\mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_t \right](\omega) = 0$, see e.g. [11, 16].

The halfspace in $L_d^p(\mathcal{F}_t)$ with normal direction $w \in L_d^q(\mathcal{F}_t) \setminus \{0\}$ is denoted by

$$G_t(w) := \left\{ u \in L_d^p(\mathcal{F}_t) : 0 \leq \mathbb{E} \left[w^\top u \right] \right\}.$$

We will define the set of dual variables to be

$$\mathcal{W}_t := \left\{ (\mathbb{Q}, w) \in \mathcal{M}_d(\mathbb{P}) \times \left(((M_t)_+)^+ \setminus M_t^\perp \right) : \tilde{w}_t^T(\mathbb{Q}, w) \in L_d^q(\mathcal{F}_T)_+ \right\}$$

where for any $0 \leq t \leq \tau \leq T$

$$\tilde{w}_t^\tau(\mathbb{Q}, w) = \text{diag}(w) \text{diag} \left(\mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_t \right] \right)^{-1} \mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_\tau \right],$$

$M_t^\perp = \{v \in L_d^q(\mathcal{F}_t) : \forall u \in M_t : \mathbb{E}[v^\top u] = 0\}$ and $C^+ = \{v \in L_d^q(\mathcal{F}_t) : \forall u \in C : \mathbb{E}[v^\top u] \geq 0\}$ denotes the positive dual cone of a cone $C \subseteq L_d^p(\mathcal{F}_t)$ for any time t .

In the following we review the definition of penalty functions and the duality results from [16].

Definition 2.3. A function $-\alpha_t : \mathcal{W}_t \rightarrow \mathcal{G}(M_t; (M_t)_+)$ is a **penalty function** at time t if it satisfies

1. $\cap_{(\mathbb{Q}, w) \in \mathcal{W}_t} -\alpha_t(\mathbb{Q}, w) \neq \emptyset$ and $-\alpha_t(\mathbb{Q}, w) \neq M_t$ for at least one $(\mathbb{Q}, w) \in \mathcal{W}_t$ and
2. $-\alpha_t(\mathbb{Q}, w) = \text{cl}(-\alpha_t(\mathbb{Q}, w) + G_t(w)) \cap M_t$ for all $(\mathbb{Q}, w) \in \mathcal{W}_t$.

Theorem 2.4 (Theorem 4.7 in [16]). A function $R_t : L_d^p(\mathcal{F}_T) \rightarrow \mathcal{G}(M_t; (M_t)_+)$ is a **closed convex conditional risk measure** if and only if there is a penalty function $-\alpha_t$ at time t such that

$$R_t(X) = \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t} \left[-\alpha_t(\mathbb{Q}, w) + \left(\mathbb{E}^\mathbb{Q}[-X | \mathcal{F}_t] + G_t(w) \right) \cap M_t \right]. \quad (2.1)$$

In particular, for R_t with the aforementioned properties, equation (2.1) is satisfied with the minimal penalty function $-\alpha_t^{\min}$ defined by

$$-\alpha_t^{\min}(\mathbb{Q}, w) = \text{cl} \bigcup_{Z \in A_t} \left(\mathbb{E}^\mathbb{Q}[Z | \mathcal{F}_t] + G_t(w) \right) \cap M_t. \quad (2.2)$$

The penalty function $-\alpha_t^{\min}$ has the property that for any penalty function $-\alpha_t$ satisfying equation (2.1) it holds that $-\alpha_t(\mathbb{Q}, w) \supseteq -\alpha_t^{\min}(\mathbb{Q}, w)$ for all $(\mathbb{Q}, w) \in \mathcal{W}_t$.

Corollary 2.5 (Corollary 4.8 in [16]). A function $R_t : L_d^p(\mathcal{F}_T) \rightarrow \mathcal{G}(M_t; (M_t)_+)$ is a **closed coherent conditional risk measure** if and only if there is a nonempty set $\mathcal{W}_{t, R_t} \subseteq \mathcal{W}_t$ such that

$$R_t(X) = \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_{t, R_t}} \left(\mathbb{E}^\mathbb{Q}[-X | \mathcal{F}_t] + G_t(w) \right) \cap M_t. \quad (2.3)$$

In particular, equation (2.3) is satisfied with \mathcal{W}_{t, R_t} replaced by \mathcal{W}_t^{\max} with

$$\mathcal{W}_t^{\max} = \{(\mathbb{Q}, w) \in \mathcal{W}_t : \tilde{w}_t^T(\mathbb{Q}, w) \in A_t^+\}.$$

If \mathcal{W}_{t, R_t} satisfies equation (2.3) then the inclusion $\mathcal{W}_{t, R_t} \subseteq \mathcal{W}_t^{\max}$ holds.

2.2 Multi-portfolio time consistency

In [16] it was shown that a useful concept of time consistency for set-valued risk measures is given by a property called multi-portfolio time consistency. In the following we review the definition and equivalent characterizations of this property.

Definition 2.6. A dynamic risk measure $(R_t)_{t=0}^T$ is called **multi-portfolio time consistent** if for all times $t \in \{0, 1, \dots, T-1\}$ and all sets $A, B \subseteq L_d^p(\mathcal{F}_T)$ the implication

$$\bigcup_{X \in A} R_{t+1}(X) \subseteq \bigcup_{Y \in B} R_{t+1}(Y) \Rightarrow \bigcup_{X \in A} R_t(X) \subseteq \bigcup_{Y \in B} R_t(Y) \quad (2.4)$$

is satisfied.

The intuitive reasoning for multi-portfolio time consistency is if at some future time the risk of a market sector A , i.e. a collection of portfolios, is greater than the risk of another market sector B then at any prior time the same relation should hold true. This property is stronger than (set-valued) time consistency, which is defined by

$$R_{t+1}(X) \subseteq R_{t+1}(Y) \Rightarrow R_t(X) \subseteq R_t(Y)$$

for any time $t \in \{0, 1, \dots, T-1\}$ and any portfolios $X, Y \in L_d^p(\mathcal{F}_T)$. However, in the scalar case, both concepts coincide with the traditional concept of time consistency.

Theorem 2.7 (Theorem 3.4 in [16]). *For a normalized dynamic risk measure $(R_t)_{t=0}^T$ the following are equivalent:*

1. $(R_t)_{t=0}^T$ is multi-portfolio time consistent,
2. R_t is recursive, that is for every time $t \in \{0, 1, \dots, T-1\}$

$$R_t(X) = \bigcup_{Z \in R_{t+1}(X)} R_t(-Z) =: R_t(-R_{t+1}(X)). \quad (2.5)$$

If additionally $M_t \subseteq M_{t+1}$ for every time $t \in \{0, 1, \dots, T-1\}$ then all of the above is also equivalent to

3. for every time $t \in \{0, 1, \dots, T-1\}$

$$A_t = A_{t+1} + A_{t,t+1}^{M_{t+1}}. \quad (2.6)$$

The above theorem provides the equivalence between multi-portfolio time consistency and the recursive form for set-valued risk measures. In [16] it was demonstrated that the set of superhedging portfolios satisfies the recursive form, but the set-valued average value at risk does not. Furthermore, [16] shows that the algorithm for calculating the set of superhedging portfolios in [32] is a result of the recursive form, which demonstrates that the recursive form can be seen as a set-valued version of Bellman's principle.

3 Convex risk measures and multi-portfolio time consistency

In this section, we want to study the impact of the multi-portfolio time consistency property on the penalty function of a closed convex risk measure. In the scalar case it could be shown that (multi-portfolio) time consistency is equivalent to an additive property of the penalty functions, see e.g. [17, 10, 8, 9, 1], which is called the cocycle property in [8, 9]. We will show that a corresponding result is also true in the set-valued case. However, it is much harder to prove than in the scalar case. The reason is that, when following the proofs along the lines of [17, 9], an additional infimum (that is the union in the recursion) appears in the set-valued case, which is not present in the scalar case. One would need to apply a minimax theorem in order to exchange

the infimum and the supremum, but it is hard to verify the constraint qualification. Thus, we will follow a different route in proving the main theorem about the equivalence between multi-portfolio time consistency and an additive property of the penalty functions. In the heart of this new proof lies a Hahn-Banach separation argument, which we will provide before presenting the main theorem.

The Hahn-Banach argument uses the functions $F_{(Y,v)}^{M_t} : L_d^p(\mathcal{F}_T) \rightarrow 2^{M_t}$ defined by

$$F_{(Y,v)}^{M_t}[X] := \left\{ u \in M_t : \mathbb{E} \left[X^\top Y \right] \leq \mathbb{E} \left[v^\top u \right] \right\},$$

for $Y \in L_d^q(\mathcal{F}_T)$, $v \in L_d^q(\mathcal{F}_t)$. These functions are the main ingredients in the duality theory for set-valued functions (see [22], example 2 and proposition 6), as they replace the continuous linear functions used in the scalar duality theory. Lemma 4.5 in [16] relates the variables (Y, v) to the dual variables $(\mathbb{Q}, w) \in \mathcal{W}_t$ used for risk measures in the following way. For every $(Y, v) \in L_d^q(\mathcal{F}_T)_+ \times (\mathbb{E}[Y|\mathcal{F}_t] + M_t^\perp) \setminus M_t^\perp$ there exists a $(\mathbb{Q}, w) \in \mathcal{W}_t$ (and vice versa) such that $F_{(Y,v)}^{M_t} = \tilde{F}_{(\mathbb{Q},w)}^{M_t}$, where

$$\tilde{F}_{(\mathbb{Q},w)}^{M_t}[X] = \left\{ u \in M_t : \mathbb{E} \left[w^\top \mathbb{E}^\mathbb{Q} [X | \mathcal{F}_t] \right] \leq \mathbb{E} \left[w^\top u \right] \right\} = \left(\mathbb{E}^\mathbb{Q} [X | \mathcal{F}_t] + G_t(w) \right) \cap M_t.$$

Clearly, the functions $\tilde{F}_{(\mathbb{Q},w)}^{M_t}$ appear in the dual representation (2.1) of risk measures and in the definition of the minimal penalty function (2.2).

For the remainder of the paper, we will assume that the same portfolios are eligible through time in all states of the market.

Assumption 3.1. Let the space of eligible portfolios M_t be such that $\tilde{M}_t = M_0$ almost surely for all times t for some subspace M_0 of \mathbb{R}^d .

We are now ready to formulate the Hahn-Banach argument, which characterizes when a portfolio is acceptable.

Lemma 3.2. Let $A_t \subseteq L_d^p(\mathcal{F}_T)$ be a closed convex acceptance set, and let $X \in L_d^p(\mathcal{F}_T)$. Then, $X \notin A_t$ if and only if there exists a $(\mathbb{Q}, w) \in \mathcal{W}_0$ such that

$$\tilde{F}_{(\mathbb{Q},w)}^{M_0}[X] \not\supseteq \text{cl} \bigcup_{Z \in A_t} \tilde{F}_{(\mathbb{Q},w)}^{M_0}[Z].$$

Proof. By the separating hyperplane theorem, if $X \notin A_t$ then there exists a $Y \in L_d^q(\mathcal{F}_T)_+$ such that $\mathbb{E}[Y^\top X] < \inf_{Z \in A_t} \mathbb{E}[Y^\top Z]$. (If we choose $Y \notin L_d^q(\mathcal{F}_T)_+$ it follows that $\inf_{Z \in A_t} \mathbb{E}[Y^\top Z] = -\infty$ by $A_t + L_d^p(\mathcal{F}_T)_+ \subseteq A_t$ which leads to a contradiction.) This implies that $F_{(Y,v)}^{M_0}[X] = \{u \in M_0 : \mathbb{E}[Y^\top X] \leq v^\top u\} \not\supseteq \{u \in M_0 : \inf_{Z \in A_t} \mathbb{E}[Y^\top Z] \leq v^\top u\} = \text{cl} \bigcup_{Z \in A_t} F_{(Y,v)}^{M_0}[Z]$ for any $v \notin M_0^\perp$ since $f(u) = v^\top u$ is a continuous linear operator from M_0 to \mathbb{R} . In particular this is true for any $v \in (\mathbb{E}[Y] + M_0^\perp) \setminus M_0^\perp$. Then by lemma 4.5 in [16] there exists a pair $(\mathbb{Q}, w) \in \mathcal{W}_0$ such that $\tilde{F}_{(\mathbb{Q},w)}^{M_0}[\cdot] = F_{(Y,v)}^{M_0}[\cdot]$, therefore $\tilde{F}_{(\mathbb{Q},w)}^{M_0}[X] \not\supseteq \text{cl} \bigcup_{Z \in A_t} \tilde{F}_{(\mathbb{Q},w)}^{M_0}[Z]$.

If $\tilde{F}_{(\mathbb{Q},w)}^{M_0}[X] \not\supseteq \text{cl} \bigcup_{Z \in A_t} \tilde{F}_{(\mathbb{Q},w)}^{M_0}[Z]$ for some $(\mathbb{Q}, w) \in \mathcal{W}_0$, then $\mathbb{E}^\mathbb{Q}[X] \neq \mathbb{E}^\mathbb{Q}[Z]$ for all $Z \in A_t$. Therefore $X \notin A_t$. \square

In order to formulate the additive property of the penalty functions, we need to define the minimal stepped penalty function $-\alpha_{t,\tau}^{\min}$ (stepped from t to $\tau > t$). The definition is straight forward, using the definition of minimal penalty functions (2.2), but with stepped acceptance sets. Define $-\alpha_{t,\tau}^{\min}$ by

$$-\alpha_{t,\tau}^{\min}(\mathbb{Q}, w) := \text{cl} \bigcup_{X \in A_{t,\tau}^{M_\tau}} \left(\mathbb{E}^\mathbb{Q} [X | \mathcal{F}_t] + G_t(w) \right) \cap M_t \quad (3.1)$$

for $(\mathbb{Q}, w) \in \mathcal{W}_{t,\tau} = \{(\mathbb{Q}, w) \in \mathcal{M}_d(\mathbb{P}) \times (((M_t)_+)^+ \setminus M_t^\perp) : \tilde{w}_t^\tau(\mathbb{Q}, w) \in ((M_\tau)_+)^+\}$. A detailed discussion about stepped risk measures can be found in section 8.3 of the appendix.

We now state the main theorem of this section. Its proof is based on the Hahn-Banach argument given above and several lemmas provided in the appendix, sections 8.1 and 8.2, that concern e.g. the relation of dual variables at different times.

Theorem 3.3. *Let assumption 3.1 be satisfied. Let $(R_t)_{t=0}^T$ be a dynamic normalized closed convex risk measure. Then, $(R_t)_{t=0}^T$ is multi-portfolio time consistent if and only if*

$$-\alpha_t^{\min}(\mathbb{Q}, w) = \text{cl} \left(-\alpha_{t,\tau}^{\min}(\mathbb{Q}, w) + \mathbb{E}^\mathbb{Q} \left[-\alpha_\tau^{\min}(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \mid \mathcal{F}_t \right] \right)$$

holds for every $(\mathbb{Q}, w) \in \mathcal{W}_t$ for all $t, \tau \in \{0, 1, \dots, T\}$ with $t < \tau$.

Proof. From theorem 2.7, a normalized dynamic risk measure $(R_t)_{t=0}^T$ with $M_t \subseteq M_{t+1}$ is multi-portfolio time consistent if and only if $A_t = A_{t,\tau}^{M_\tau} + A_\tau$ for every $t, \tau \in \{0, 1, \dots, T\}$ with $t < \tau$.

1. Assume $(R_t)_{t=0}^T$ is a normalized convex multi-portfolio time consistent risk measure, i.e. assume $A_t = A_{t,\tau}^{M_\tau} + A_\tau$.

First, let $A_t \subseteq A_{t,\tau}^{M_\tau} + A_\tau$, that is for any $X \in A_t$ there exists $X_{t,\tau} \in A_{t,\tau}^{M_\tau}$ and $X_\tau \in A_\tau$ such that $X = X_{t,\tau} + X_\tau$. Let $(\mathbb{Q}, w) \in \mathcal{W}_t$ (then $(\mathbb{Q}, w) \in \mathcal{W}_{t,\tau}$, see remark 8.8 and $(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in \mathcal{W}_\tau$ by lemma 8.1). Then using proposition 8.4

$$\begin{aligned} \left(\mathbb{E}^\mathbb{Q} [X \mid \mathcal{F}_t] + G_t(w) \right) \cap M_t &= \left(\mathbb{E}^\mathbb{Q} [X_{t,\tau} + X_\tau \mid \mathcal{F}_t] + G_t(w) \right) \cap M_t \\ &= \left(\mathbb{E}^\mathbb{Q} [X_{t,\tau} \mid \mathcal{F}_t] + G_t(w) \right) \cap M_t \\ &\quad + \mathbb{E}^\mathbb{Q} \left[\left(\mathbb{E}^\mathbb{Q} [X_\tau \mid \mathcal{F}_\tau] + G_\tau(\tilde{w}_t^\tau(\mathbb{Q}, w)) \right) \cap M_\tau \mid \mathcal{F}_t \right] \\ &\subseteq -\alpha_{t,\tau}^{\min}(\mathbb{Q}, w) + \mathbb{E}^\mathbb{Q} \left[-\alpha_\tau^{\min}(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \mid \mathcal{F}_t \right]. \end{aligned}$$

This implies

$$\begin{aligned} -\alpha_t^{\min}(\mathbb{Q}, w) &= \text{cl} \bigcup_{X \in A_t} \left(\mathbb{E}^\mathbb{Q} [X \mid \mathcal{F}_t] + G_t(w) \right) \cap M_t \\ &\subseteq \text{cl} \left(-\alpha_{t,\tau}^{\min}(\mathbb{Q}, w) + \mathbb{E}^\mathbb{Q} \left[-\alpha_\tau^{\min}(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \mid \mathcal{F}_t \right] \right). \end{aligned}$$

Second, let $A_t \supseteq A_{t,\tau}^{M_\tau} + A_\tau$, that is for any $X_{t,\tau} \in A_{t,\tau}^{M_\tau}$ and $X_\tau \in A_\tau$ then $X_{t,\tau} + X_\tau \in A_t$. Let $(\mathbb{Q}, w) \in \mathcal{W}_t$ (then $(\mathbb{Q}, w) \in \mathcal{W}_{t,\tau}$ as mentioned in remark 8.8 and $(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in \mathcal{W}_\tau$ by lemma 8.1). Then by proposition 8.4

$$\begin{aligned} \left(\mathbb{E}^\mathbb{Q} [X_{t,\tau} \mid \mathcal{F}_t] + G_t(w) \right) \cap M_t + \mathbb{E}^\mathbb{Q} \left[\left(\mathbb{E}^\mathbb{Q} [X_\tau \mid \mathcal{F}_\tau] + G_\tau(\tilde{w}_t^\tau(\mathbb{Q}, w)) \right) \cap M_\tau \mid \mathcal{F}_t \right] \\ = \left(\mathbb{E}^\mathbb{Q} [X_{t,\tau} + X_\tau \mid \mathcal{F}_t] + G_t(w) \right) \cap M_t \\ \subseteq -\alpha_t^{\min}(\mathbb{Q}, w). \end{aligned} \tag{3.2}$$

Therefore

$$\begin{aligned} &\text{cl} \left(-\alpha_{t,\tau}^{\min}(\mathbb{Q}, w) + \mathbb{E}^\mathbb{Q} \left[-\alpha_\tau^{\min}(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \mid \mathcal{F}_t \right] \right) \\ &= \text{cl} \left(-\alpha_{t,\tau}^{\min}(\mathbb{Q}, w) + \text{cl} \bigcup_{X_\tau \in A_\tau} \left(\mathbb{E}^\mathbb{Q} [X_\tau \mid \mathcal{F}_\tau] + G_\tau(w) \right) \cap M_\tau \right) \\ &= \text{cl} \left(\text{cl} \bigcup_{X_{t,\tau} \in A_{t,\tau}^{M_\tau}} \left(\mathbb{E}^\mathbb{Q} [X_{t,\tau} \mid \mathcal{F}_t] + G_t(w) \right) \cap M_t + \text{cl} \bigcup_{X_\tau \in A_\tau} \left(\mathbb{E}^\mathbb{Q} [X_\tau \mid \mathcal{F}_\tau] + G_\tau(w) \right) \cap M_\tau \right) \end{aligned} \tag{3.3}$$

$$\begin{aligned}
&= \text{cl} \left(\bigcup_{X_{t,\tau} \in A_{t,\tau}^{M_\tau}} \left(\mathbb{E}^\mathbb{Q} [X_{t,\tau} | \mathcal{F}_t] + G_t(w) \right) \cap M_t + \bigcup_{X_\tau \in A_\tau} \left(\mathbb{E}^\mathbb{Q} [X_\tau | \mathcal{F}_t] + G_t(w) \right) \cap M_t \right) \\
&= \text{cl} \bigcup_{\substack{X_{t,\tau} \in A_{t,\tau}^{M_\tau} \\ X_\tau \in A_\tau}} \left(\mathbb{E}^\mathbb{Q} [X_{t,\tau} + X_\tau | \mathcal{F}_t] + G_t(w) \right) \cap M_t \\
&\subseteq -\alpha_t^{\min}(\mathbb{Q}, w).
\end{aligned} \tag{3.4}$$

Equation (3.3) follows from lemma 8.3, and equation (3.4) follows from proposition 1.23 in [31].

2. Conversely, assume $-\alpha_t^{\min}(\mathbb{Q}, w) = \text{cl} \left(-\alpha_{t,\tau}^{\min}(\mathbb{Q}, w) + \mathbb{E}^\mathbb{Q} [-\alpha_\tau^{\min}(\mathbb{Q}, \tilde{w}_t^r(\mathbb{Q}, w)) | \mathcal{F}_t] \right)$ for every $(\mathbb{Q}, w) \in \mathcal{W}_t$.

Note that for any $(\mathbb{Q}, w) \in \mathcal{W}_0$ it holds $\tilde{w}_0^\tau(\mathbb{Q}, w) = \tilde{w}_t^r(\mathbb{Q}, \tilde{w}_0^t(\mathbb{Q}, w))$.

Let $X \in A_{t,\tau}^{M_\tau} + A_\tau$, then from equation (3.2) and proposition 8.4, for every $(\mathbb{Q}, w) \in \mathcal{W}_0$

$$\begin{aligned}
\tilde{F}_{(\mathbb{Q},w)}^{M_0} [X] &\subseteq \mathbb{E}^\mathbb{Q} \left[\text{cl} \left(-\alpha_{t,\tau}^{\min}(\mathbb{Q}, \tilde{w}_0^t(\mathbb{Q}, w)) + \mathbb{E}^\mathbb{Q} [-\alpha_\tau^{\min}(\mathbb{Q}, \tilde{w}_0^\tau(\mathbb{Q}, w)) | \mathcal{F}_t] \right) \right] \\
&= \mathbb{E}^\mathbb{Q} [-\alpha_t^{\min}(\mathbb{Q}, \tilde{w}_0^t(\mathbb{Q}, w))] \\
&= \text{cl} \bigcup_{Z \in A_t} \tilde{F}_{(\mathbb{Q},w)}^{M_0} [Z].
\end{aligned}$$

The last equality follows from lemma 8.3. If we assume that $X \notin A_t$ then, by lemma 3.2, there exists a pair $(\mathbb{Q}, w) \in \mathcal{W}_0$ such that $\tilde{F}_{(\mathbb{Q},w)}^{M_0} [X] \not\supseteq \text{cl} \bigcup_{Z \in A_t} \tilde{F}_{(\mathbb{Q},w)}^{M_0} [Z]$. However, this is a contradiction to the above, therefore $X \in A_t$ and thus

$$A_{t,\tau}^{M_\tau} + A_\tau \subseteq A_t. \tag{3.5}$$

Let $X \in A_t$, then (using proposition 8.4 and lemma 8.3)) for every $(\mathbb{Q}, w) \in \mathcal{W}_0$

$$\begin{aligned}
\tilde{F}_{(\mathbb{Q},w)}^{M_0} [X] &\subseteq \mathbb{E}^\mathbb{Q} [-\alpha_t^{\min}(\mathbb{Q}, \tilde{w}_0^t(\mathbb{Q}, w))] \\
&= \mathbb{E}^\mathbb{Q} \left[\text{cl} \left(-\alpha_{t,\tau}^{\min}(\mathbb{Q}, \tilde{w}_0^t(\mathbb{Q}, w)) + \mathbb{E}^\mathbb{Q} [-\alpha_\tau^{\min}(\mathbb{Q}, \tilde{w}_0^\tau(\mathbb{Q}, w)) | \mathcal{F}_t] \right) \right] \\
&= \text{cl} \bigcup_{Z \in A_{t,\tau}^{M_\tau} + A_\tau} \tilde{F}_{(\mathbb{Q},w)}^{M_0} [Z].
\end{aligned}$$

If we assume that $X \notin A_{t,\tau}^{M_\tau} + A_\tau$ (which is a closed convex acceptance set by lemma 8.6 and where the assumption $A_{t,\tau}^{M_\tau} + A_\tau \subseteq A_t$ of lemma 8.6 is satisfied by (3.5)) then, by lemma 3.2, there exists a pair $(\mathbb{Q}, w) \in \mathcal{W}_0$ such that

$$\tilde{F}_{(\mathbb{Q},w)}^{M_0} [X] \not\supseteq \text{cl} \bigcup_{Z \in A_{t,\tau}^{M_\tau} + A_\tau} \tilde{F}_{(\mathbb{Q},w)}^{M_0} [Z].$$

This is a contradiction to the above, therefore $X \in A_{t,\tau}^{M_\tau} + A_\tau$.

□

In the above theorem we have demonstrated that the sum of penalty functions gives an equivalent characterization of multi-portfolio time consistency. This allows us to define risk measures by the penalty functions alone and verify whether the corresponding closed convex risk measure is multi-portfolio time consistent.

4 Coherent risk measures and multi-portfolio time consistency

In this section, we want to study multi-portfolio time consistency in the coherent case. In particular, we want to find equivalent characterizations of multi-portfolio time consistency with respect to the set of dual variables. In the scalar framework an equivalent property is given by stability of the dual variables, also called m-stability, which was studied for the case when the dual probability measures are absolutely continuous to the real world probability measure \mathbb{P} in [1, 10], and when the dual probability measures are equivalent to \mathbb{P} in [13, 17, 4].

For the results below we use the definition of the maximal set of stepped dual variables $\mathcal{W}_{t,\tau}^{\max} \subseteq \mathcal{W}_{t,\tau}$ as defined in section 8.3. That is,

$$\mathcal{W}_{t,\tau}^{\max} = \{(\mathbb{Q}, w) \in \mathcal{W}_{t,\tau} : -\alpha_{t,\tau}^{\min}(\mathbb{Q}, w) = G_t(w) \cap M_t\} = \left\{(\mathbb{Q}, w) \in \mathcal{W}_{t,\tau} : \tilde{w}_t^\tau(\mathbb{Q}, w) \in (A_{t,\tau}^{M_\tau})^+\right\}.$$

Remark 4.1. For any closed coherent risk measure R_t (not necessarily multi-portfolio time consistent) under assumption 3.1 it can trivially be seen that $\mathcal{W}_t^{\max} \subseteq \mathcal{W}_{t,\tau}^{\max}$ since $\mathcal{W}_t \subseteq \mathcal{W}_{t,\tau}$ (see remark 8.8) and $\tilde{w}_t^T(\mathbb{Q}, w) \in A_t^+$ implies $\tilde{w}_t^\tau(\mathbb{Q}, w) \in (A_{t,\tau}^{M_\tau})^+$.

The first result we provide, which will be useful for generating a closed coherent multi-portfolio time consistent risk measure in section 5, is a corollary to theorem 3.3 above.

Let us define the set $H_t^\tau : 2^{\mathcal{W}_\tau} \rightarrow 2^{\mathcal{W}_t}$ by

$$H_t^\tau(D) := \{(\mathbb{Q}, w) \in \mathcal{W}_t : (\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in D\}$$

for any $0 \leq t \leq \tau$, and any $D \subseteq \mathcal{W}_\tau$.

Corollary 4.2. *Let assumption 3.1 be satisfied. Let $(R_t)_{t=0}^T$ be a dynamic normalized closed coherent risk measure. Then, $(R_t)_{t=0}^T$ is multi-portfolio time consistent if and only if for all $t, \tau \in \{0, 1, \dots, T\}$ with $t < \tau$ it holds*

$$\mathcal{W}_t^{\max} = \mathcal{W}_{t,\tau}^{\max} \cap H_t^\tau(\mathcal{W}_\tau^{\max}).$$

Proof. This follows trivially from theorem 3.3 and proposition 8.4 by noting that for any times t and $\tau > t$

$$\begin{aligned} -\alpha_t^{\min}(\mathbb{Q}, w) &= \begin{cases} G_t(w) \cap M_t & \text{if } (\mathbb{Q}, w) \in \mathcal{W}_t^{\max} \\ M_t & \text{else} \end{cases} \\ -\alpha_{t,\tau}^{\min}(\mathbb{Q}, w) &= \begin{cases} G_t(w) \cap M_t & \text{if } (\mathbb{Q}, w) \in \mathcal{W}_{t,\tau}^{\max} \\ M_t & \text{else.} \end{cases} \end{aligned}$$

And since $\mathcal{W}_{t,\tau} \supseteq \mathcal{W}_t$ (see remark 8.8) for any times $t < \tau$, we have $\mathcal{W}_{t,\tau}^{\max} \cap H_t^\tau(\mathcal{W}_\tau^{\max}) = (\mathcal{W}_{t,\tau}^{\max} \cap \mathcal{W}_t) \cap H_t^\tau(\mathcal{W}_\tau^{\max})$, therefore the result follows. \square

We now want to study the pasting of dual variables and the generalization of stability to the set-valued case.

For $\mathbb{Q}, \mathbb{R} \in \mathcal{M}_d(\mathbb{P})$ we denote by $\mathbb{Q} \oplus^\tau \mathbb{R}$ the pasting of \mathbb{Q} and \mathbb{R} in τ , i.e. the vector probability measures $\mathbb{S} \in \mathcal{M}_d(\mathbb{P})$ defined via

$$\frac{d\mathbb{S}}{d\mathbb{P}} = \text{diag} \left(\mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_\tau \right] \right) \text{diag} \left(\mathbb{E} \left[\frac{d\mathbb{R}}{d\mathbb{P}} \middle| \mathcal{F}_\tau \right] \right)^{-1} \frac{d\mathbb{R}}{d\mathbb{P}}.$$

In the set-valued framework we will define stability as a property with respect to two other sets. This is due to the fact that our dual variables consists of pairs. Naturally, stability is a property that imposes conditions on both components of a pair (\mathbb{Q}, w) .

Definition 4.3. A set $W_t \subseteq \mathcal{W}_t$ is called **stable** at time t with respect to $W_{t,\tau}$ and W_τ for $\tau > t$ if

1. $(\mathbb{Q}, w) \in W_t$ implies $(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in W_\tau$ and
2. $(\mathbb{Q}, w) \in W_{t,\tau}$ and $\mathbb{R} \in \mathcal{M}_d(\mathbb{P})$ such that $(\mathbb{R}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in W_\tau$ implies $(\mathbb{Q} \oplus^\tau \mathbb{R}, w) \in W_t$.

The main theorem of this section is given below. It provides an equivalence between the stability of the sets of dual variables \mathcal{W}_t^{\max} and multi-portfolio time consistency. We present an additional property which is equivalent to stability and therefore to multi-portfolio time consistency. This additional property, given in equation (4.1), is a generalization of property (2) of corollary 1.26 from [1].

Theorem 4.4. Let assumption 3.1 be satisfied. Let $(R_t)_{t=0}^T$ be a normalized closed coherent risk measure, then the following are equivalent:

1. $(R_t)_{t=0}^T$ is multi-portfolio time consistent;
2. \mathcal{W}_t^{\max} is stable at time t with respect to $\mathcal{W}_{t,\tau}^{\max}$ and \mathcal{W}_τ^{\max} for every time $t, \tau \in \{0, 1, \dots, T\}$ with $t < \tau$;
3. for every time $t, \tau \in \{0, 1, \dots, T\}$ with $t < \tau$

$$\mathcal{W}_t^{\max} = \{(\mathbb{Q} \oplus^\tau \mathbb{R}, w) : (\mathbb{Q}, w) \in \mathcal{W}_{t,\tau}^{\max}, (\mathbb{R}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in \mathcal{W}_\tau^{\max}\}. \quad (4.1)$$

Proof. We will show that multi-portfolio time consistency implies stability, stability implies equation (4.1), and finally, that equation (4.1) implies multi-portfolio time consistency.

1. \Rightarrow 2. Assume $(R_t)_{t=0}^T$ is multi-portfolio time consistent. We want to show that \mathcal{W}_t^{\max} is stable at time t with respect to $\mathcal{W}_{t,\tau}^{\max}$ and \mathcal{W}_τ^{\max} , as given in definition 4.3.

- (a) By corollary 4.2 it follows that $\mathcal{W}_t^{\max} \subseteq H_t^\tau(\mathcal{W}_\tau^{\max})$ and thus $(\mathbb{Q}, w) \in \mathcal{W}_t^{\max}$ implies $(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in \mathcal{W}_\tau^{\max}$.
- (b) Let $t < \tau$, $(\mathbb{Q}, w) \in \mathcal{W}_{t,\tau}^{\max}$ and $\mathbb{R} \in \mathcal{M}_d(\mathbb{P})$ with $(\mathbb{R}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in \mathcal{W}_\tau^{\max}$. We need to show that $(\mathbb{S}, w) \in \mathcal{W}_t^{\max}$ where $\mathbb{S} = \mathbb{Q} \oplus^\tau \mathbb{R}$ is the pasting of \mathbb{Q} and \mathbb{R} at time τ .
 - i. $(\mathbb{S}, w) \in \mathcal{W}_t$ since $\mathbb{S} \in \mathcal{M}_d(\mathbb{P})$, $w \in ((M_t)_+)^+ \setminus M_t^\perp$, and

$$\tilde{w}_t^T(\mathbb{S}, w) = \tilde{w}_\tau^T(\mathbb{R}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in L_d^q(\mathcal{F}_T)_+.$$

- ii. $(\mathbb{S}, w) \in \mathcal{W}_t^{\max}$ if $-\alpha_t^{\min}(\mathbb{S}, w) = G_t(w) \cap M_t$. By theorem 3.3 it follows that for any $(\mathbb{S}, w) \in \mathcal{W}_t$

$$-\alpha_t^{\min}(\mathbb{S}, w) = \text{cl} \left(-\alpha_{t,\tau}^{\min}(\mathbb{S}, w) + \mathbb{E}^\mathbb{S} [-\alpha_\tau^{\min}(\mathbb{S}, \tilde{w}_t^\tau(\mathbb{S}, w)) | \mathcal{F}_t] \right).$$

It holds $-\alpha_{t,\tau}^{\min}(\mathbb{S}, w) = -\alpha_{t,\tau}^{\min}(\mathbb{Q}, w)$ by the tower property, and $-\alpha_\tau^{\min}(\mathbb{S}, \tilde{w}_t^\tau(\mathbb{S}, w)) = -\alpha_\tau^{\min}(\mathbb{R}, \tilde{w}_t^\tau(\mathbb{Q}, w))$ by

$$\begin{aligned} -\alpha_\tau^{\min}(\mathbb{S}, \tilde{w}_t^\tau(\mathbb{S}, w)) &= \text{cl} \bigcup_{X \in A_\tau} \left(\mathbb{E}^\mathbb{S} [X | \mathcal{F}_\tau] + G_\tau(\tilde{w}_t^\tau(\mathbb{S}, w)) \right) \cap M_\tau \\ &= \text{cl} \bigcup_{X \in A_\tau} \left\{ x \in M_\tau : \mathbb{E} \left[\tilde{w}_t^\tau(\mathbb{S}, w)^\top \mathbb{E}^\mathbb{S} [X | \mathcal{F}_\tau] \right] \leq \mathbb{E} \left[\tilde{w}_t^\tau(\mathbb{S}, w)^\top x \right] \right\} \\ &= \text{cl} \bigcup_{X \in A_\tau} \left\{ x \in M_\tau : \mathbb{E} \left[\tilde{w}_t^T(\mathbb{S}, w)^\top X \right] \leq \mathbb{E} \left[\tilde{w}_t^\tau(\mathbb{Q}, w)^\top x \right] \right\} \\ &= \text{cl} \bigcup_{X \in A_\tau} \left\{ x \in M_\tau : \mathbb{E} \left[\tilde{w}_\tau^T(\mathbb{R}, \tilde{w}_t^\tau(\mathbb{Q}, w))^\top X \right] \leq \mathbb{E} \left[\tilde{w}_t^\tau(\mathbb{Q}, w)^\top x \right] \right\} \end{aligned}$$

$$\begin{aligned}
&= \text{cl} \bigcup_{X \in A_\tau} \left\{ x \in M_\tau : \mathbb{E} \left[\tilde{w}_t^\tau(\mathbb{Q}, w)^\top \mathbb{E}^\mathbb{R} [X | \mathcal{F}_\tau] \right] \leq \mathbb{E} \left[\tilde{w}_t^\tau(\mathbb{Q}, w)^\top x \right] \right\} \\
&= -\alpha_\tau^{\min}(\mathbb{R}, \tilde{w}_t^\tau(\mathbb{Q}, w)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
-\alpha_t^{\min}(\mathbb{S}, w) &= \text{cl} \left(-\alpha_{t,\tau}^{\min}(\mathbb{S}, w) + \mathbb{E}^\mathbb{S} \left[-\alpha_\tau^{\min}(\mathbb{S}, \tilde{w}_t^\tau(\mathbb{S}, w)) \mid \mathcal{F}_t \right] \right) \\
&= \text{cl} \left(-\alpha_{t,\tau}^{\min}(\mathbb{Q}, w) + \mathbb{E}^\mathbb{Q} \left[-\alpha_\tau^{\min}(\mathbb{R}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \mid \mathcal{F}_t \right] \right) \\
&= \text{cl} \left(G_t(w) \cap M_t + \mathbb{E}^\mathbb{Q} \left[G_\tau(\tilde{w}_t^\tau(\mathbb{Q}, w)) \cap M_\tau \mid \mathcal{F}_t \right] \right) \\
&= G_t(w) \cap M_t,
\end{aligned}$$

using $(\mathbb{Q}, w) \in \mathcal{W}_{t,\tau}^{\max}$ and $(\mathbb{R}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in \mathcal{W}_\tau^{\max}$. The last line follows from proposition 8.4.

Therefore for every time t , \mathcal{W}_t^{\max} is stable at time t with respect to $\mathcal{W}_{t,\tau}^{\max}$ and \mathcal{W}_τ^{\max} for every $\tau > t$.

2. \Rightarrow 3. We will demonstrate that stability implies equation (4.1). If for every time t , \mathcal{W}_t^{\max} is stable at time t with respect to $\mathcal{W}_{t,\tau}^{\max}$ and \mathcal{W}_τ^{\max} for every $\tau > t$ then trivially “ \supseteq ” in equation (4.1) follows by the second property of stability. By the first property of stability and remark 4.1 then for any $(\mathbb{Q}, w) \in \mathcal{W}_t^{\max}$ it follows that $(\mathbb{Q}, w) \in \mathcal{W}_{t,\tau}^{\max}$ and $(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in \mathcal{W}_\tau^{\max}$. Since $\mathbb{Q} = \mathbb{Q} \oplus^\tau \mathbb{Q}$ for any time τ and any probability measure $\mathbb{Q} \in \mathcal{M}_d(\mathbb{P})$, then “ \subseteq ” in equation (4.1) trivially follows.

3. \Rightarrow 1. We will prove that equation (4.1) implies that for every $(\mathbb{Q}, w) \in \mathcal{W}_t$ it holds $-\alpha_t^{\min}(\mathbb{Q}, w) = \text{cl} \left(-\alpha_{t,\tau}^{\min}(\mathbb{Q}, w) + \mathbb{E}^\mathbb{Q} \left[-\alpha_\tau^{\min}(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \mid \mathcal{F}_t \right] \right)$ which in turn implies multi-portfolio time consistency by theorem 3.3.

(a) We will show that $\mathcal{W}_t^{\max} \subseteq \{(\mathbb{Q} \oplus^\tau \mathbb{R}, w) : (\mathbb{Q}, w) \in \mathcal{W}_{t,\tau}^{\max}, (\mathbb{R}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in \mathcal{W}_\tau^{\max}\}$ implies $-\alpha_t^{\min}(\mathbb{Q}, w) \supseteq \text{cl} \left(-\alpha_{t,\tau}^{\min}(\mathbb{Q}, w) + \mathbb{E}^\mathbb{Q} \left[-\alpha_\tau^{\min}(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \mid \mathcal{F}_t \right] \right)$ for every $(\mathbb{Q}, w) \in \mathcal{W}_t$.

i. Let $(\mathbb{S}, w) \in \mathcal{W}_t^{\max}$. Then, $-\alpha_t^{\min}(\mathbb{S}, w) = G_t(w) \cap M_t$. Additionally, there exists a $\mathbb{Q}, \mathbb{R} \in \mathcal{M}_d(\mathbb{P})$ such that $(\mathbb{Q}, w) \in \mathcal{W}_{t,\tau}^{\max}$, $(\mathbb{R}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in \mathcal{W}_\tau^{\max}$, and $\mathbb{S} = \mathbb{Q} \oplus^\tau \mathbb{R}$. This implies $-\alpha_{t,\tau}^{\min}(\mathbb{S}, w) = -\alpha_{t,\tau}^{\min}(\mathbb{Q}, w) = G_t(w) \cap M_t$ and $-\alpha_\tau^{\min}(\mathbb{S}, \tilde{w}_t^\tau(\mathbb{S}, w)) = -\alpha_\tau^{\min}(\mathbb{R}, \tilde{w}_t^\tau(\mathbb{Q}, w)) = G_\tau(\tilde{w}_t^\tau(\mathbb{Q}, w)) \cap M_\tau = G_\tau(\tilde{w}_t^\tau(\mathbb{S}, w)) \cap M_\tau$. Therefore, using proposition 8.4,

$$\begin{aligned}
-\alpha_t^{\min}(\mathbb{S}, w) &= G_t(w) \cap M_t \\
&= \text{cl} \left(G_t(w) \cap M_t + \mathbb{E}^\mathbb{S} \left[G_\tau(\tilde{w}_t^\tau(\mathbb{S}, w)) \cap M_\tau \mid \mathcal{F}_t \right] \right) \\
&= \text{cl} \left(-\alpha_{t,\tau}^{\min}(\mathbb{S}, w) + \mathbb{E}^\mathbb{S} \left[-\alpha_\tau^{\min}(\mathbb{S}, \tilde{w}_t^\tau(\mathbb{S}, w)) \mid \mathcal{F}_t \right] \right).
\end{aligned}$$

ii. Let $(\mathbb{S}, w) \in \mathcal{W}_t \setminus \mathcal{W}_t^{\max}$, then $-\alpha_t^{\min}(\mathbb{S}, w) = M_t$. Therefore it automatically follows that $-\alpha_t^{\min}(\mathbb{S}, w) \supseteq \text{cl} \left(-\alpha_{t,\tau}^{\min}(\mathbb{S}, w) + \mathbb{E}^\mathbb{S} \left[-\alpha_\tau^{\min}(\mathbb{S}, \tilde{w}_t^\tau(\mathbb{S}, w)) \mid \mathcal{F}_t \right] \right)$.

(b) We will show that $\mathcal{W}_t^{\max} \supseteq \{(\mathbb{Q} \oplus^\tau \mathbb{R}, w) : (\mathbb{Q}, w) \in \mathcal{W}_{t,\tau}^{\max}, (\mathbb{R}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in \mathcal{W}_\tau^{\max}\}$ implies $-\alpha_t^{\min}(\mathbb{Q}, w) \subseteq \text{cl} \left(-\alpha_{t,\tau}^{\min}(\mathbb{Q}, w) + \mathbb{E}^\mathbb{Q} \left[-\alpha_\tau^{\min}(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \mid \mathcal{F}_t \right] \right)$ for every $(\mathbb{Q}, w) \in \mathcal{W}_t$.

i. Let $(\mathbb{S}, w) \in \{(\mathbb{Q} \oplus^\tau \mathbb{R}, w) : (\mathbb{Q}, w) \in \mathcal{W}_{t,\tau}^{\max}, (\mathbb{R}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in \mathcal{W}_\tau^{\max}\}$. Then by assumption $(\mathbb{S}, w) \in \mathcal{W}_t^{\max} \subseteq \mathcal{W}_t$ and $-\alpha_t^{\min}(\mathbb{S}, w) = G_t(w) \cap M_t$. Additionally, there exists some $\mathbb{Q}, \mathbb{R} \in \mathcal{M}_d(\mathbb{P})$ such that $(\mathbb{Q}, w) \in \mathcal{W}_{t,\tau}^{\max}$, $(\mathbb{R}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in \mathcal{W}_\tau^{\max}$.

- \mathcal{W}_τ^{\max} , and $\mathbb{S} = \mathbb{Q} \oplus^\tau \mathbb{R}$. As above this implies that $-\alpha_{t,\tau}^{\min}(\mathbb{S}, w) = G_t(w) \cap M_t$ and $-\alpha_\tau^{\min}(\mathbb{S}, \tilde{w}_t^\tau(\mathbb{S}, w)) = G_\tau(\tilde{w}_t^\tau(\mathbb{S}, w)) \cap M_\tau$. Therefore $-\alpha_t^{\min}(\mathbb{S}, w) = \text{cl}(-\alpha_{t,\tau}^{\min}(\mathbb{S}, w) + \mathbb{E}^\mathbb{S}[-\alpha_\tau^{\min}(\mathbb{S}, \tilde{w}_t^\tau(\mathbb{S}, w)) | \mathcal{F}_t])$.
- ii. Let $(\mathbb{S}, w) \in \mathcal{W}_t \setminus \{(\mathbb{Q} \oplus^\tau \mathbb{R}, w) : (\mathbb{Q}, w) \in \mathcal{W}_{t,\tau}^{\max}, (\mathbb{R}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in \mathcal{W}_\tau^{\max}\}$. Then for every $\mathbb{Q}, \mathbb{R} \in \mathcal{M}_d(\mathbb{P})$ such that $\mathbb{S} = \mathbb{Q} \oplus^\tau \mathbb{R}$ either $(\mathbb{Q}, w) \notin \mathcal{W}_{t,\tau}^{\max}$ or $(\mathbb{R}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \notin \mathcal{W}_\tau^{\max}$. This implies for any $\mathbb{Q}, \mathbb{R} \in \mathcal{M}_d(\mathbb{P})$ where $\mathbb{S} = \mathbb{Q} \oplus^\tau \mathbb{R}$ either $-\alpha_{t,\tau}^{\min}(\mathbb{S}, w) = -\alpha_{t,\tau}^{\min}(\mathbb{Q}, w) = M_t$ or $-\alpha_\tau^{\min}(\mathbb{S}, \tilde{w}_t^\tau(\mathbb{S}, w)) = -\alpha_\tau^{\min}(\mathbb{R}, \tilde{w}_t^\tau(\mathbb{Q}, w)) = M_\tau$. Therefore $\text{cl}(-\alpha_{t,\tau}^{\min}(\mathbb{S}, w) + \mathbb{E}^\mathbb{S}[-\alpha_\tau^{\min}(\mathbb{S}, \tilde{w}_t^\tau(\mathbb{S}, w)) | \mathcal{F}_t]) = M_t$. Thus we have $-\alpha_t^{\min}(\mathbb{S}, w) \subseteq \text{cl}(-\alpha_{t,\tau}^{\min}(\mathbb{S}, w) + \mathbb{E}^\mathbb{S}[-\alpha_\tau^{\min}(\mathbb{S}, \tilde{w}_t^\tau(\mathbb{S}, w)) | \mathcal{F}_t])$.

□

The above theorem provides two equivalent representations for multi-portfolio time consistency for coherent risk measures. This generalizes the stability property for scalar risk measures, which is a well known result. Conceptually, stability means that pasting together dual variables creates another possible dual variable, which logically corresponds with time consistency concepts.

5 Composition of one-step risk measures

As in section 2.1 in [11] and section 4 in [12], a (multi-portfolio) time consistent version of any scalar dynamic risk measure can be created through backwards recursion. In the following we recall the corresponding results from proposition 3.10 and corollary 3.13 in [16] in the set-valued framework. Then, in corollary 5.3, we prove an equivalent formulation for closed convex and coherent risk measures, which will be very useful to deduce dual representations of composed (and thus multi-portfolio time consistent) risks measures.

Proposition 5.1. *Let $(R_t)_{t=0}^T$ be a dynamic risk measure on $L_d^p(\mathcal{F}_T)$ and let $M_t \subseteq M_{t+1}$ for every time $t \in \{0, 1, \dots, T-1\}$, then $(\tilde{R}_t)_{t=0}^T$ defined for all $X \in L_d^p(\mathcal{F}_T)$ by*

$$\tilde{R}_T(X) = R_T(X), \quad (5.1)$$

$$\forall t \in \{0, 1, \dots, T-1\} : \tilde{R}_t(X) = \bigcup_{Z \in \tilde{R}_{t+1}(X)} R_t(-Z) \quad (5.2)$$

is multi-portfolio time consistent. Furthermore, $(\tilde{R}_t)_{t=0}^T$ is M_t -translative and satisfies monotonicity, but may fail to be finite at zero. Additionally, if $(R_t)_{t=0}^T$ is convex (coherent, closed) then $(\tilde{R}_t)_{t=0}^T$ is convex (coherent, closed).

Proof. All but the closedness property was proven in proposition 3.10 in [16].

\tilde{R}_t is closed if and only if \tilde{A}_t is a closed set. Using corollary 3.13 in [16] (cited below) we have in particular, $\tilde{A}_T = A_T$ is closed. Then using backwards induction we will assume \tilde{A}_{t+1} is closed and need to show that $\tilde{A}_t = A_{t,t+1}^{M_{t+1}} + \tilde{A}_{t+1}$ is a closed set. An inspection of lemma 8.5 shows that its proof is analog when replacing A_{t+1} by \tilde{A}_{t+1} as long as \tilde{A}_{t+1} is closed. Thus, it follows that \tilde{A}_t is closed. □

Corollary 5.2 (Corollary 3.13 in [16]). *Let $M_t \subseteq M_{t+1}$ for every time $t \in \{0, 1, \dots, T-1\}$. Let $(R_t)_{t=0}^T$ be a dynamic risk measure on $L_d^p(\mathcal{F}_T)$ with acceptance sets $(A_t)_{t=0}^T$. Then, the following are equivalent:*

1. $(\tilde{R}_t)_{t=0}^T$ is defined as in equations (5.1) and (5.2);

2. $(\tilde{A}_t)_{t=0}^T$ is defined by

$$\begin{aligned}\tilde{A}_T &= A_T, \\ \forall t \in \{0, 1, \dots, T-1\} : \tilde{A}_t &= A_{t,t+1}^{M_{t+1}} + \tilde{A}_{t+1},\end{aligned}$$

where $(\tilde{A}_t)_{t=0}^T$ denotes the acceptance set of $(\tilde{R}_t)_{t=0}^T$.

Corollary 5.3. *Let the assumptions of corollary 5.2 and assumption 3.1 be satisfied and let additionally*

3. $(R_t)_{t=0}^T$ be closed and convex with minimal penalty function $(-\alpha_t^{\min})_{t=0}^T$. Then, $(-\tilde{\alpha}_t^{\min})_{t=0}^T$ defined recursively by

$$\begin{aligned}-\tilde{\alpha}_T^{\min}(\mathbb{Q}_T, w_T) &= -\alpha_T^{\min}(\mathbb{Q}_T, w_T), \\ \forall t \in \{0, 1, \dots, T-1\} : -\tilde{\alpha}_t^{\min}(\mathbb{Q}_t, w_t) &= \text{cl} \left(-\alpha_{t,t+1}^{\min}(\mathbb{Q}_t, w_t) + \mathbb{E}^{\mathbb{Q}} \left[-\tilde{\alpha}_{t+1}^{\min}(\mathbb{Q}_t, \tilde{w}_t^{t+1}(\mathbb{Q}_t, w_t)) \mid \mathcal{F}_t \right] \right)\end{aligned}$$

for every $(\mathbb{Q}_t, w_t) \in \mathcal{W}_t$, is equivalently defined by

$$-\tilde{\alpha}_t^{\min}(\mathbb{Q}, w) := \text{cl} \bigcup_{Z \in \tilde{A}_t} \left(\mathbb{E}^{\mathbb{Q}}[Z \mid \mathcal{F}_t] + G_t(w) \right) \cap M_t, \quad (5.3)$$

where $(\tilde{A}_t)_{t=0}^T$ is obtained by the recursion in property 2 in corollary 5.2. The dynamic risk measure $(\tilde{R}_t)_{t=0}^T$ corresponding to $(\tilde{A}_t)_{t=0}^T$ is closed convex and multi-portfolio time consistent (but may fail to be finite at zero). Further, if \tilde{R}_t is finite at zero then \tilde{R}_t is equivalent to its dual form with penalty function $-\tilde{\alpha}_t^{\min}$.

4. $(R_t)_{t=0}^T$ be closed and coherent (and thus normalized) with maximal dual set $(\mathcal{W}_t^{\max})_{t=0}^T$. Then, $(\tilde{\mathcal{W}}_t^{\max})_{t=0}^T$ defined recursively by

$$\begin{aligned}\tilde{\mathcal{W}}_T^{\max} &= \mathcal{W}_T^{\max}, \\ \forall t \in \{0, 1, \dots, T-1\} : \tilde{\mathcal{W}}_t^{\max} &= \mathcal{W}_{t,t+1}^{\max} \cap H_t^{t+1}(\tilde{\mathcal{W}}_{t+1}^{\max}),\end{aligned}$$

is equivalently defined by

$$\tilde{\mathcal{W}}_t^{\max} := \left\{ (\mathbb{Q}, w) \in \mathcal{W}_t : \tilde{w}_t^T(\mathbb{Q}, w) \in \tilde{A}_t^+ \right\},$$

where $(\tilde{A}_t)_{t=0}^T$ is obtained by the recursion in property 2 in corollary 5.2. The dynamic risk measure $(\tilde{R}_t)_{t=0}^T$ corresponding to $(\tilde{A}_t)_{t=0}^T$ is closed coherent and multi-portfolio time consistent, and is finite at zero if and only if $\tilde{\mathcal{W}}_t^{\max} \neq \emptyset$ for all times t .

Proof. 3.: The proof of theorem 3.3 demonstrates the equivalence between the sum of penalty functions and the sum of acceptance sets, where A and $-\alpha$ have to be replaced by \tilde{A} and $-\tilde{\alpha}$ at the appropriate places. Regarding the assumptions of theorem 3.3: closure and convexity follow from proposition 5.1 and normalization is not needed for this equivalence as stated in remark 4 in [16]. Notice that lemma 3.2 does not require the finite at zero properties for acceptance sets. Finally, if \tilde{R}_t is finite at zero, then it is equivalent to its dual representation with minimal penalty function $-\tilde{\alpha}_t^{\min}$ by theorem 2.4.

4.: Using the definition in (5.3), corollary 4.2, where \mathcal{W}, A and $-\alpha$ is replaced by $\tilde{\mathcal{W}}, \tilde{A}$ and $-\tilde{\alpha}$ at the appropriate places, yields the equivalence between the two definitions of $(\tilde{\mathcal{W}}_t^{\max})_{t=0}^T$. Closure and coherence follow from proposition 5.1 and normalization is not needed for this equivalence as stated in remark 4 in [16]. Additionally, $\tilde{R}_t(0) \neq \emptyset$, see

proof of proposition 3.11 in [16]. Furthermore, $\tilde{R}_t(0) \neq M_t$ implies that \tilde{R}_t is proper and thus the dual representation holds true. Then, there exists a $(\mathbb{Q}, w) \in \mathcal{W}_t$ such that $-\tilde{\alpha}_t^{\min}(\mathbb{Q}, w) \neq M_t$, i.e. $\tilde{\mathcal{W}}_t^{\max} \neq \emptyset$. And $\tilde{R}_t(0) = M_t$ implies, by proposition 13 (iv) in [22], $M_t = \tilde{R}_t(0) \subseteq \tilde{R}_t^{**}(0) = \bigcap_{(\mathbb{Q}, w) \in \tilde{\mathcal{W}}_t^{\max}} G_t(w) \cap M_t$ and thus $\tilde{\mathcal{W}}_t^{\max} = \emptyset$. \square

The set-valued average value at risk was shown not to be multi-portfolio time consistent in [16] (and similarly the scalar average value at risk is well known to not be time consistent). In section 6.2 we will use corollary 5.3 to construct the composed version of the average value at risk and deduce its dual representation.

6 Examples

6.1 Superhedging

In this section, we show that the dynamic superhedging portfolios in markets with proportional transaction costs satisfy the stability condition for the dual variables. The set of superhedging portfolios in markets with proportional transaction costs were studied in [29, 36, 30, 24, 32]. Its dynamic extension was given in [16] and shown to be multi-portfolio time consistent.

Let us consider a market with proportional transaction costs as in [29, 36, 30], which is modeled by a sequence of solvency cones $(K_t)_{t=0}^T$. K_t is a solvency cone at time t if it is an \mathcal{F}_t -measurable cone such that for every $\omega \in \Omega$, $K_t(\omega)$ is a closed convex cone with $\mathbb{R}_+^d \subseteq K_t(\omega) \subsetneq \mathbb{R}^d$. K_t can be generated by the time t bid-ask exchange rates between any two assets, for details see [29, 36, 30]. The solvency cone at time t can be interpreted as the set of positions which can be exchanged into a nonnegative portfolio at time t by trading according to the prevailing bid-ask exchange rates. Let us denote by

$$C_{t,T} := - \sum_{s=t}^T L_d^p(\mathcal{F}_s; K_s).$$

$C_{t,T}$ can be interpreted as the set of $L_d^p(\mathcal{F}_T)$ -valued random portfolios $V_T : \Omega \rightarrow \mathbb{R}^d$ that can be reached by trading self-financingly until time T starting with 0 endowment at time t , see [29, 36, 30, 16].

As shown in [16], the set of superhedging portfolios at time t is given by

$$SHP_t(X) := \{u \in L_d^p(\mathcal{F}_t) : -X + u \in -C_{t,T}\}.$$

If the market process $(K_t)_{t=0}^T$ satisfies an appropriate no arbitrage condition (called robust no arbitrage, see [29, 36, 30, 16] for details), then the set of superhedging portfolios has the following dual representation

$$SHP_t(X) = \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_{\{t, \dots, T\}}} \left(\mathbb{E}^{\mathbb{Q}}[X | \mathcal{F}_t] + G_t(w) \right), \quad (6.1)$$

where $t, \tau \in \{0, \dots, T\}$ with $t < \tau$ and

$$\begin{aligned} \mathcal{W}_{\{t, \dots, \tau\}} &:= \{(\mathbb{Q}, w) \in \mathcal{W}_t^K : \tilde{w}_t^s(\mathbb{Q}, w) \in L_d^q(\mathcal{F}_s; K_s^+) \forall s \in \{t, \dots, \tau\}, \\ &\quad \tilde{w}_t^\tau(\mathbb{Q}, w) \in L_d^q(\mathcal{F}_\tau; (L_d^p(\mathcal{F}_\tau) \cap \sum_{s=\tau+1}^T L_d^p(\mathcal{F}_s; K_s))^+) \}; \\ \mathcal{W}_t^K &:= \{(\mathbb{Q}, w) \in \mathcal{M}_d(\mathbb{P}) \times (L_d^q(\mathcal{F}_t; K_t^+) \setminus \{0\}) : \tilde{w}_t^T(\mathbb{Q}, w) \in L_d^q(\mathcal{F}_T)_+\} \subseteq \mathcal{W}_t. \end{aligned}$$

Since the stepped acceptance set is defined by $A_{t,\tau} = -C_{t,T} \cap L_d^p(\mathcal{F}_\tau)$, the logic of corollary 8.11, and property (v) on page 7 in [37], the stepped superhedging portfolios can be defined by the set of dual variables $\mathcal{W}_{\{t,\dots,\tau\}} \subseteq \mathcal{W}_{t,\tau} = \mathcal{W}_t$ (where $\mathcal{W}_{t,\tau} = \mathcal{W}_t$ by $M_t = L_d^p(\mathcal{F}_t)$ for all times t and remark 8.9).

It was further demonstrated in [16] that the set-valued function given by $R_t(X) := SHP_t(-X)$ defines a dynamic risk measure which is normalized, closed, coherent, and multi-portfolio time consistent. Thus, in order to show that the dual variables satisfy stability, it would remain to show that $(\mathcal{W}_{\{t,\dots,T\}})_{t=0}^T$ satisfies equation (4.1).

Lemma 6.1. *For any time t and any $\tau > t$,*

$$\mathcal{W}_{\{t,\dots,T\}} = \{(\mathbb{Q} \oplus^\tau \mathbb{R}, w) : (\mathbb{Q}, w) \in \mathcal{W}_{\{t,\dots,\tau\}}, (\mathbb{R}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in \mathcal{W}_{\{\tau,\dots,T\}}\}.$$

Proof. Let us show that $\mathcal{W}_{\{t,\dots,T\}} \subseteq \{(\mathbb{Q} \oplus^\tau \mathbb{R}, w) : (\mathbb{Q}, w) \in \mathcal{W}_{\{t,\dots,\tau\}}, (\mathbb{R}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in \mathcal{W}_{\{\tau,\dots,T\}}\}$. Take $(\mathbb{Q}, w) \in \mathcal{W}_{\{t,\dots,T\}}$. Then it trivially follows that $(\mathbb{Q}, w) \in \mathcal{W}_{\{t,\dots,\tau\}}$. Additionally, it holds $(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in \mathcal{W}_\tau^K$ and $\tilde{w}_\tau^s(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) = \tilde{w}_t^s(\mathbb{Q}, w)$ for every $s \geq \tau$. Therefore, it follows that $\tilde{w}_\tau^s(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in L_d^q(\mathcal{F}_s; K_s^+)$ for every $s \geq \tau$ and thus we have $(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in \mathcal{W}_{\{\tau,\dots,T\}}$. And since $\mathbb{Q} \oplus^\tau \mathbb{Q} = \mathbb{Q}$, we have shown “ \subseteq ”.

Conversely, if $(\mathbb{Q}, w) \in \mathcal{W}_{\{t,\dots,\tau\}}$ and $(\mathbb{R}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in \mathcal{W}_{\{\tau,\dots,T\}}$ then letting $\mathbb{S} = \mathbb{Q} \oplus^\tau \mathbb{R}$ we wish to show that $(\mathbb{S}, w) \in \mathcal{W}_{\{t,\dots,T\}}$. For any $s \in \{t, \dots, \tau\}$ it follows that $\mathbb{E} \left[\frac{d\mathbb{S}}{d\mathbb{P}} \middle| \mathcal{F}_s \right] = \mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_s \right]$, and thus $\tilde{w}_t^s(\mathbb{S}, w) = \tilde{w}_t^s(\mathbb{Q}, w) \in L_d^q(\mathcal{F}_s; K_s^+)$. If $s \in \{\tau + 1, \dots, T\}$ then

$$\begin{aligned} \tilde{w}_t^s(\mathbb{S}, w) &= \text{diag}(w) \text{diag} \left(\mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_t \right] \right)^{-1} \text{diag} \left(\mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_\tau \right] \right) \text{diag} \left(\mathbb{E} \left[\frac{d\mathbb{R}}{d\mathbb{P}} \middle| \mathcal{F}_\tau \right] \right)^{-1} \mathbb{E} \left[\frac{d\mathbb{R}}{d\mathbb{P}} \middle| \mathcal{F}_s \right] \\ &= \text{diag}(\tilde{w}_t^\tau(\mathbb{Q}, w)) \text{diag} \left(\mathbb{E} \left[\frac{d\mathbb{R}}{d\mathbb{P}} \middle| \mathcal{F}_\tau \right] \right)^{-1} \mathbb{E} \left[\frac{d\mathbb{R}}{d\mathbb{P}} \middle| \mathcal{F}_s \right] \\ &= \tilde{w}_\tau^s(\mathbb{R}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in L_d^q(\mathcal{F}_s; K_s^+). \end{aligned}$$

Finally, $(\mathbb{S}, w) \in \mathcal{W}_t^K$ trivially follows from the same logic as above. Therefore $\mathcal{W}_{\{t,\dots,T\}} \supseteq \{(\mathbb{Q} \oplus^\tau \mathbb{R}, w) : (\mathbb{Q}, w) \in \mathcal{W}_{\{t,\dots,\tau\}}, (\mathbb{R}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in \mathcal{W}_{\{\tau,\dots,T\}}\}$ \square

In the frictionless case the no arbitrage condition implies $A_{t,t+1} = A_t \cap L_d^p(\mathcal{F}_{t+1}) = L_d^p(\mathcal{F}_t; K_t) + L_d^p(\mathcal{F}_{t+1}; K_{t+1})$ (see e.g. section 4.2 in [33]), and thus the stepped dual variables can be simplified to

$$\mathcal{W}_{\{t,\dots,\tau\}} = \{(\mathbb{Q}, w) \in \mathcal{W}_t^K : \tilde{w}_t^s(\mathbb{Q}, w) \in L_d^q(\mathcal{F}_s; K_s^+) \forall s \in \{t, \dots, \tau\}\}.$$

Since the proof of lemma 6.1 above does not require the additional condition that $\tilde{w}_t^\tau(\mathbb{Q}, w) \in L_d^q(\mathcal{F}_\tau; (L_d^p(\mathcal{F}_\tau) \cap \sum_{s=\tau+1}^T L_d^p(\mathcal{F}_s; K_s))^+)$ for any $(\mathbb{Q}, w) \in \mathcal{W}_{\{t,\dots,\tau\}}$, we can immediately conclude that dual variables for the frictionless superhedging price are stable as well.

6.2 Average Value at Risk

In this section we will discuss the dynamic set-valued average value at risk and the composed dynamic set-valued average value at risk. As the underlying spaces we consider $L_d^p(\mathcal{F}_t), L_d^q(\mathcal{F}_t)$ with $p = 1$ and $q = \infty$.

The dual definition for the dynamic average value at risk with time t parameters $\lambda^t \in L_d^\infty(\mathcal{F}_t)$ where $0 < \lambda_i^t < 1$ is given by

$$AV @ R_t^\lambda(X) := \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t^\lambda} \left(\mathbb{E}^\mathbb{Q}[-X | \mathcal{F}_t] + G_t(w) \right) \cap M_t \quad (6.2)$$

for any $X \in L_d^1(\mathcal{F}_T)$ where

$$\mathcal{W}_t^\lambda := \left\{ (\mathbb{Q}, w) \in \mathcal{W}_t : \text{diag}(w) \left(\text{diag}(\lambda^t)^{-1} \bar{\mathbf{1}} - \text{diag} \left(\mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_t \right] \right)^{-1} \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \in L_d^\infty(\mathcal{F}_T)_+ \right\}$$

and $\bar{\mathbf{1}} := (1, \dots, 1)^\top \in \mathbb{R}^d$, see section 5.2 in [16].

In the following proposition, we deduce the acceptance set and thus the primal representation for the dynamic average value at risk given in (6.2). This proves that (6.2) is the dynamic version of the closure of the static average value at risk defined via its acceptance set A_0^λ in [26].

Proposition 6.2. *The acceptance set associated with the conditional average value at risk at time t and parameter λ^t is given by $\bar{A}_t^\lambda = \text{cl}(A_t^\lambda)$ where*

$$A_t^\lambda = \left\{ X \in L_d^1(\mathcal{F}_T) : \exists Z \in L_d^1(\mathcal{F}_T)_+ : X + Z - \text{diag}(\lambda^t)^{-1} \mathbb{E}[Z | \mathcal{F}_t] \in L_d^1(\mathcal{F}_T)_+ \right\}$$

and \mathcal{W}_t^λ is the maximal dual set.

Proof. By corollary 2.5, in order to show that \bar{A}_t^λ is the acceptance set for $AV @ R_t^\lambda$ and \mathcal{W}_t^λ is the maximal dual set we need to verify that $\mathcal{W}_t^\lambda = \{(\mathbb{Q}, w) \in \mathcal{W}_t : \tilde{w}_t^T(\mathbb{Q}, w) \in (A_t^\lambda)^+\}$, since $(A_t^\lambda)^+ = (\bar{A}_t^\lambda)^+$.

It can easily be seen that $A_t^\lambda = \bigcup_{Z \in L_d^1(\mathcal{F}_T)_+} \left(\text{diag}(\lambda^t)^{-1} \mathbb{E}[Z | \mathcal{F}_t] - Z \right) + L_d^1(\mathcal{F}_T)_+$. Additionally, $\bigcup_{Z \in L_d^1(\mathcal{F}_T)_+} \left(\text{diag}(\lambda^t)^{-1} \mathbb{E}[Z | \mathcal{F}_t] - Z \right)$ is a $(\mathcal{F}_t\text{-conditional})$ cone containing 0 (letting $Z = 0$). Therefore $\tilde{w}_t^T(\mathbb{Q}, w) \in (A_t^\lambda)^+$ if and only if

$$\tilde{w}_t^T(\mathbb{Q}, w) \in \left(\bigcup_{Z \in L_d^1(\mathcal{F}_T)_+} \left(\text{diag}(\lambda^t)^{-1} \mathbb{E}[Z | \mathcal{F}_t] - Z \right) \right)^+ \cap L_d^\infty(\mathcal{F}_T)_+$$

by property (v) on page 7 in [37]. By $(\mathbb{Q}, w) \in \mathcal{W}_t$, it already follows that $\tilde{w}_t^T(\mathbb{Q}, w) \in L_d^\infty(\mathcal{F}_T)_+$. Thus the maximal dual set for \bar{A}_t^λ is given by

$$\left\{ (\mathbb{Q}, w) \in \mathcal{W}_t : \tilde{w}_t^T(\mathbb{Q}, w) \in \left(\bigcup_{Z \in L_d^1(\mathcal{F}_T)_+} \left(\text{diag}(\lambda^t)^{-1} \mathbb{E}[Z | \mathcal{F}_t] - Z \right) \right)^+ \right\}.$$

The condition $\tilde{w}_t^T(\mathbb{Q}, w) \in \left(\bigcup_{Z \in L_d^1(\mathcal{F}_T)_+} \left(\text{diag}(\lambda^t)^{-1} \mathbb{E}[Z | \mathcal{F}_t] - Z \right) \right)^+$ is true if and only if for every $Z \in L_d^1(\mathcal{F}_T)_+$

$$\begin{aligned} 0 &\leq \mathbb{E} \left[\tilde{w}_t^T(\mathbb{Q}, w)^\top \left(\text{diag}(\lambda^t)^{-1} \mathbb{E}[Z | \mathcal{F}_t] - Z \right) \right] \\ &= \mathbb{E} \left[\tilde{w}_t^T(\mathbb{Q}, w)^\top \text{diag}(\lambda^t)^{-1} \mathbb{E}[Z | \mathcal{F}_t] \right] - \mathbb{E} \left[\tilde{w}_t^T(\mathbb{Q}, w)^\top Z \right] \\ &= \mathbb{E} \left[\left(\text{diag}(\lambda^t)^{-1} w \right)^\top Z \right] - \mathbb{E} \left[\tilde{w}_t^T(\mathbb{Q}, w)^\top Z \right] \\ &= \mathbb{E} \left[\left(\text{diag}(\lambda^t)^{-1} w - \tilde{w}_t^T(\mathbb{Q}, w) \right)^\top Z \right]. \end{aligned}$$

This means that this additional condition is equivalent to

$$\text{diag}(\lambda^t)^{-1} w - \tilde{w}_t^T(\mathbb{Q}, w) = \text{diag}(w) \left(\text{diag}(\lambda^t)^{-1} \bar{\mathbf{1}} - \text{diag} \left(\mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_t \right] \right)^{-1} \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \in L_d^\infty(\mathcal{F}_T)_+.$$

Therefore $\mathcal{W}_t^\lambda = \{(\mathbb{Q}, w) \in \mathcal{W}_t : \tilde{w}_t^T(\mathbb{Q}, w) \in (A_t^\lambda)^+\}$. \square

In proposition 5.4 in [16] it was shown that $(AV@R_t^\lambda)_{t=0}^T$ is a normalized closed coherent dynamic risk measure. It can easily be seen that $(AV@R_t^\lambda)_{t=0}^T$ is not a multi-portfolio time consistent risk measure. However, by backward recursion we can construct a composed, multi-portfolio time consistent version of the set-valued average value at risk and corollary 5.3 provides a method to find its dual representation. In the scalar case the composed average value at risk is studied in [11].

Lemma 6.3. *Let assumption 3.1 be satisfied. The composed version of the average value at risk $(AV@R_t^\lambda)_{t=0}^T$ is given by*

$$\widetilde{AV@R_t^\lambda}(X) := \bigcap_{(\mathbb{Q}, w) \in \widetilde{\mathcal{W}}_t^\lambda} \left(\mathbb{E}^\mathbb{Q}[-X | \mathcal{F}_t] + G_t(w) \right) \cap M_t,$$

where

$$\begin{aligned} \widetilde{\mathcal{W}}_t^\lambda := & \{ (\mathbb{Q}, w) \in \mathcal{W}_t : \forall \tau \in \{t, \dots, T-1\} : \forall Z \in L_d^1(\mathcal{F}_{\tau+1})_+ : \\ & \mathbb{E} \left[\left(\text{diag}(\tilde{w}_t^\tau(\mathbb{Q}, w)) \left(\text{diag}(\lambda^\tau)^{-1} \vec{1} - \text{diag} \left(\mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_\tau \right] \right)^{-1} \mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_{\tau+1} \right] \right) \right)^\top Z \right] \geq \\ & \sup \left\{ \mathbb{E} \left[\tilde{w}_t^{\tau+1}(\mathbb{Q}, w)^\top D \right] : D \in L_d^1(\mathcal{F}_{\tau+1})_- \cap \left[M_{\tau+1} + \left(\text{diag}(\lambda^\tau)^{-1} \mathbb{E}[Z | \mathcal{F}_\tau] - Z \right) \right] \right\} \}. \end{aligned}$$

$(\widetilde{AV@R_t^\lambda})_{t=0}^T$ is a multi-portfolio time consistent risk measure.

Remark 6.4. The dual representation in lemma 6.3 simplifies a lot if all assets are eligible, i.e., if $M_t = L_d^1(\mathcal{F}_t)$ for all times t . Then, the composed average value at risk at time t is defined by the dual set

$$\begin{aligned} \widetilde{\mathcal{W}}_t^\lambda = & \{ (\mathbb{Q}, w) \in \mathcal{W}_t : \forall \tau \in \{t, \dots, T-1\} : \\ & \text{diag}(\tilde{w}_t^\tau(\mathbb{Q}, w)) \left(\text{diag}(\lambda^\tau)^{-1} \vec{1} - \text{diag} \left(\mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_\tau \right] \right)^{-1} \mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_{\tau+1} \right] \right) \in L_d^1(\mathcal{F}_{\tau+1})_+ \}. \end{aligned}$$

This follows since by lemma 6.5, the stepped average value at risk has in this case maximal dual sets

$$\mathcal{W}_{t,\tau}^\lambda = \left\{ (\mathbb{Q}, w) \in \mathcal{W}_t : \text{diag}(w) \left(\text{diag}(\lambda^t)^{-1} \vec{1} - \text{diag} \left(\mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_t \right] \right)^{-1} \mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_\tau \right] \right) \in L_d^\infty(\mathcal{F}_\tau)_+ \right\}$$

for all times $t, \tau \in \{0, 1, \dots, T\}$ with $t < \tau$, where $\mathcal{W}_{t,\tau} = \mathcal{W}_t$ by remark 8.9. This dual representation can be interpreted as the extension of the stepped scalar representation given in [11].

Proof of lemma 6.3. By corollary 5.3, $(\widetilde{AV@R_t^\lambda})_{t=0}^T$ is the multi-portfolio time consistent version of $(AV@R_t^\lambda)_{t=0}^T$ if and only if

$$\begin{aligned} \widetilde{\mathcal{W}}_T^\lambda &= \mathcal{W}_T^\lambda \\ \widetilde{\mathcal{W}}_t^\lambda &= H_t^{t+1} \left(\widetilde{\mathcal{W}}_{t+1}^\lambda \right) \cap \mathcal{W}_{t,t+1}^\lambda \end{aligned}$$

where $\widetilde{\mathcal{W}}_t^\lambda \neq \emptyset$ for all times t . Trivially it can be seen that $\widetilde{\mathcal{W}}_T^\lambda = \mathcal{W}_T$. Furthermore, $\mathcal{W}_T^\lambda = \mathcal{W}_T$ since $(\lambda_i^T)^{-1} - 1 \geq 0$ for every $i = 1, \dots, d$ (by $0 < \lambda_i^T < 1$) and $w = \tilde{w}_T^T(\mathbb{Q}, w) \in L_d^\infty(\mathcal{F}_T)_+$ by

$(\mathbb{Q}, w) \in \mathcal{W}_T$, and therefore the product is almost surely nonnegative. By lemma 6.5 below it holds

$$\begin{aligned} \mathcal{W}_{t,t+1}^\lambda = & \{(\mathbb{Q}, w) \in \mathcal{W}_{t,\tau} : \forall Z \in L_d^1(\mathcal{F}_{t+1})_+ : \\ & \mathbb{E} \left[\left(\text{diag}(w) \left(\text{diag}(\lambda^t)^{-1} \vec{1} - \text{diag} \left(\mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_t \right] \right)^{-1} \mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_{t+1} \right] \right) \right)^\top Z \right] \geq \\ & \sup \left\{ \mathbb{E} \left[w^\top \mathbb{E}^\mathbb{Q} [D | \mathcal{F}_t] \right] : D \in L_d^1(\mathcal{F}_{t+1})_- \cap \left[M_{t+1} + \left(\text{diag}(\lambda^t)^{-1} \mathbb{E} [Z | \mathcal{F}_t] - Z \right) \right] \right\} \}. \end{aligned}$$

Furthermore, using lemma 8.1 and $\tilde{w}_{t+1}^\tau(\mathbb{Q}, \tilde{w}_t^{t+1}(\mathbb{Q}, w)) = \tilde{w}_t^\tau(\mathbb{Q}, w)$ it follows

$$\begin{aligned} H_t^{t+1}(\tilde{\mathcal{W}}_{t+1}^\lambda) = & \{(\mathbb{Q}, w) \in \mathcal{W}_t : \forall \tau \in \{t+1, \dots, T-1\} : \forall Z \in L_d^1(\mathcal{F}_{\tau+1})_+ : \\ & \mathbb{E} \left[\left(\text{diag}(\tilde{w}_t^\tau(\mathbb{Q}, w)) \left(\text{diag}(\lambda^\tau)^{-1} \vec{1} - \text{diag} \left(\mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_\tau \right] \right)^{-1} \mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_{\tau+1} \right] \right) \right)^\top Z \right] \geq \\ & \sup \left\{ \mathbb{E} \left[\tilde{w}_t^{\tau+1}(\mathbb{Q}, w)^\top D \right] : D \in L_d^1(\mathcal{F}_{\tau+1})_- \cap \left[M_{\tau+1} + \left(\text{diag}(\lambda^\tau)^{-1} \mathbb{E} [Z | \mathcal{F}_\tau] - Z \right) \right] \right\} \}. \end{aligned}$$

Noting that $\tilde{w}_\tau^\tau(\mathbb{Q}, w) = w$ for any time τ , the recursive form $\tilde{\mathcal{W}}_t^\lambda = H_t^{t+1}(\tilde{\mathcal{W}}_{t+1}^\lambda) \cap \mathcal{W}_{t,t+1}^\lambda$ is proven. Finally, $\tilde{\mathcal{W}}_t^\lambda \neq \emptyset$ holds since $(\mathbb{P}, w) \in \tilde{\mathcal{W}}_t^\lambda$ for any $w \in L_d^\infty(\mathcal{F}_t)_+$. This is because $(\mathbb{P}, w) \in \mathcal{W}_t$ and for any $\tau \in \{t, \dots, T-1\}$

$$\mathbb{E} \left[\left(\text{diag}(w) \left(\text{diag}(\lambda^\tau)^{-1} \vec{1} - \vec{1} \right) \right)^\top Z \right] \geq 0$$

for every $Z \in L_d^1(\mathcal{F}_T)_+$ and $\mathbb{E} [w^\top D] \leq 0$ for every $D \in L_d^1(\mathcal{F}_T)_-$. \square

Lemma 6.5. *The stepped average value at risk from time t to τ (for $t, \tau \in \{0, 1, \dots, T\}$ with $t < \tau$) with time t parameters $\lambda^t \in L_d^\infty(\mathcal{F}_t)$ where $0 < \lambda_i^t < 1$ is given by*

$$AV@R_{t,\tau}^\lambda(X) := \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_{t,\tau}^\lambda} \left(\mathbb{E}^\mathbb{Q} [-X | \mathcal{F}_t] + G_t(w) \right) \cap M_t$$

for any $X \in L_d^1(\mathcal{F}_T)$ where

$$\begin{aligned} \mathcal{W}_{t,\tau}^\lambda = & \{(\mathbb{Q}, w) \in \mathcal{W}_{t,\tau} : \forall Z \in L_d^1(\mathcal{F}_\tau)_+ : \\ & \mathbb{E} \left[\left(\text{diag}(w) \left(\text{diag}(\lambda^t)^{-1} \vec{1} - \text{diag} \left(\mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_t \right] \right)^{-1} \mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_\tau \right] \right) \right)^\top Z \right] \geq \\ & \sup \left\{ \mathbb{E} \left[\tilde{w}_t^\tau(\mathbb{Q}, w)^\top D \right] : D \in L_d^1(\mathcal{F}_\tau)_- \cap \left[M_\tau + \left(\text{diag}(\lambda^t)^{-1} \mathbb{E} [Z | \mathcal{F}_t] - Z \right) \right] \right\} \} \end{aligned}$$

is the associated maximal stepped dual set.

Proof. Using the definition of the acceptance set for $AV@R_t^\lambda$ given in proposition 6.2, we find the stepped acceptance set is given by $\bar{A}_{t,\tau}^\lambda = \text{cl}(A_{t,\tau}^\lambda)$ where

$$\begin{aligned} A_{t,\tau}^\lambda = & A_t^\lambda \cap M_\tau = \left\{ X \in M_\tau : \exists Z \in L_d^1(\mathcal{F}_T)_+ : X + Z - \text{diag}(\lambda^t)^{-1} \mathbb{E} [Z | \mathcal{F}_t] \in L_d^1(\mathcal{F}_T)_+ \right\} \\ = & \left\{ X \in M_\tau : \exists Z \in L_d^1(\mathcal{F}_\tau)_+ : X + Z - \text{diag}(\lambda^t)^{-1} \mathbb{E} [Z | \mathcal{F}_t] \in L_d^1(\mathcal{F}_\tau)_+ \right\} \\ = & \left(\bigcup_{Z \in L_d^1(\mathcal{F}_\tau)_+} \left(\text{diag}(\lambda^t)^{-1} \mathbb{E} [Z | \mathcal{F}_t] - Z \right) + L_d^1(\mathcal{F}_\tau)_+ \right) \cap M_\tau. \end{aligned}$$

By corollary 8.11 and $(A_{t,\tau}^\lambda)^+ = (\bar{A}_{t,\tau}^\lambda)^+$, the maximal stepped dual set is given by

$$\left\{ (\mathbb{Q}, w) \in \mathcal{W}_{t,\tau} : \tilde{w}_t^\tau(\mathbb{Q}, w) \in (A_{t,\tau}^\lambda)^+ \right\}.$$

It can trivially be seen that $X \in A_{t,\tau}^\lambda$ if and only if $X = \text{diag}(\lambda^t)^{-1} \mathbb{E}[Z | \mathcal{F}_t] - Z + D$ for some $Z \in L_d^1(\mathcal{F}_\tau)_+$ and $D \in L_d^1(\mathcal{F}_\tau)_+ \cap \left[M_\tau + (Z - \text{diag}(\lambda^t)^{-1} \mathbb{E}[Z | \mathcal{F}_t]) \right]$. Therefore $\tilde{w}_t^\tau(\mathbb{Q}, w) \in (A_{t,\tau}^\lambda)^+$ if and only if for every $Z \in L_d^1(\mathcal{F}_\tau)_+$ and $D \in L_d^1(\mathcal{F}_\tau)_+ \cap \left[M_\tau + (Z - \text{diag}(\lambda^t)^{-1} \mathbb{E}[Z | \mathcal{F}_t]) \right]$

$$\begin{aligned} 0 &\leq \mathbb{E} \left[\tilde{w}_t^\tau(\mathbb{Q}, w)^\top \left(\text{diag}(\lambda^t)^{-1} \mathbb{E}[Z | \mathcal{F}_t] - Z + D \right) \right] \\ &= \mathbb{E} \left[\left(\text{diag}(\lambda^t)^{-1} w - \tilde{w}_t^\tau(\mathbb{Q}, w) \right)^\top Z \right] + \mathbb{E} \left[\tilde{w}_t^\tau(\mathbb{Q}, w)^\top D \right]. \end{aligned}$$

That is, for every $Z \in L_d^1(\mathcal{F}_\tau)_+$

$$\begin{aligned} &\sup \left\{ \mathbb{E} \left[\tilde{w}_t^\tau(\mathbb{Q}, w)^\top D \right] : D \in L_d^1(\mathcal{F}_\tau)_- \cap \left[M_\tau + \left(\text{diag}(\lambda^t)^{-1} \mathbb{E}[Z | \mathcal{F}_t] - Z \right) \right] \right\} \\ &\leq \mathbb{E} \left[\left(\text{diag}(w) \left(\text{diag}(\lambda^t)^{-1} \vec{1} - \text{diag} \left(\mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_t \right] \right)^{-1} \mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_\tau \right] \right) \right)^\top Z \right]. \end{aligned}$$

□

7 Conclusion

Multi-portfolio time consistency is investigated for closed convex and coherent dynamic risk measures on $L_d^p(\mathcal{F}_T)$. We show that the equivalent definitions for multi-portfolio time consistency are the generalizations of the equivalent definitions in the scalar case, thus lending support to this being the proper set-valued version of (scalar) time consistency. In the convex case we find that multi-portfolio time consistency is equivalent to a condition on the sum of minimal penalty functions, sometimes called the cocycle condition. In the coherent case we give a generalized version for the stability of the dual variables and show that this is equivalent to multi-portfolio time consistency. We then show that the dynamic superhedging portfolios in markets with transaction costs satisfy this stability condition, verifying its usefulness in practice. Finally, a rule for constructing multi-portfolio time consistent versions of convex and coherent risk measures is provided and applied to the set-valued average value at risk. This construction has a clean form for the dual variables and appears as a generalization of the time consistent version in the scalar case.

8 Appendix

8.1 On the relationship of dual variables at different times

In considering how closed convex (and coherent) risk measures relate through time we must consider how the sets of dual variables relate. In the following lemma we provide such a relationship between elements of \mathcal{W}_t and elements of \mathcal{W}_τ for any times t, τ with $t \leq \tau$. In fact we define a mapping on \mathcal{W}_t which is equivalent (in the set-valued replacement for continuous linear functionals) in \mathcal{W}_τ . In the scalar framework this type of property is not needed since the set \mathcal{W}_t can be simplified to any $\mathbb{Q} \ll \mathbb{P}$ for any time t .

Lemma 8.1. *Let assumption 3.1 be satisfied. Then for any choice of times t and $\tau > t$ it follows that:*

1. $\{(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) : (\mathbb{Q}, w) \in \mathcal{W}_t\} \subseteq \mathcal{W}_\tau$,
2. for every $(\mathbb{R}, v) \in \mathcal{W}_\tau$ there exists $(\mathbb{Q}, w) \in \mathcal{W}_t$ such that $\tilde{F}_{(\mathbb{R}, v)}^{M_\tau} = \tilde{F}_{(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w))}^{M_\tau}$.

Proof. 1. $\{(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) : (\mathbb{Q}, w) \in \mathcal{W}_t\} \subseteq \mathcal{W}_\tau$ if and only if $\tilde{w}_t^\tau(\mathbb{Q}, w) \in ((M_\tau)_+)^+ \setminus M_\tau^\perp$ and $\tilde{w}_\tau^T(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in L_d^q(\mathcal{F}_T)_+$ for all $(\mathbb{Q}, w) \in \mathcal{W}_t$.

- (a) Let $(\mathbb{Q}, w) \in \mathcal{W}_t$. Show $\tilde{w}_t^\tau(\mathbb{Q}, w) \in ((M_\tau)_+)^+ \setminus M_\tau^\perp$:
 - i. Let $m_\tau \in (M_\tau)_+$, then

$$\mathbb{E} \left[\tilde{w}_t^\tau(\mathbb{Q}, w)^\top m_\tau \right] = \mathbb{E} \left[w^\top \mathbb{E}^\mathbb{Q} [m_\tau | \mathcal{F}_t] \right] \geq 0$$

since $\mathbb{E}^\mathbb{Q} [m_\tau | \mathcal{F}_t] \in (M_t)_+$ by $M_t \supseteq M_\tau \cap L_d^p(\mathcal{F}_t)$ and $\tilde{M}_\tau[\omega_1] = \tilde{M}_\tau[\omega_2]$ for almost every $\omega_1, \omega_2 \in \Omega$.

- ii. Since $(\mathbb{Q}, w) \in \mathcal{W}_t$, in particular since $w \notin M_t^\perp$ there exists $m_t \in M_t \subseteq M_\tau$ such that $\mathbb{E} [w^\top m_t] \neq 0$. Then,

$$\mathbb{E} \left[\tilde{w}_t^\tau(\mathbb{Q}, w)^\top m_t \right] = \mathbb{E} \left[w^\top \mathbb{E}^\mathbb{Q} [m_t | \mathcal{F}_t] \right] = \mathbb{E} [w^\top m_t] \neq 0.$$

- (b) $\tilde{w}_\tau^T(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) = \tilde{w}_t^T(\mathbb{Q}, w) \in L_d^q(\mathcal{F}_T)_+$ by $(\mathbb{Q}, w) \in \mathcal{W}_t$.

2. By lemma 4.5 in [16] it follows that for every $(\mathbb{R}, v) \in \mathcal{W}_\tau$ there exists a (Y, \bar{v}) with $Y \in L_d^q(\mathcal{F}_T)_+$, $\bar{v} \in (\mathbb{E} [Y | \mathcal{F}_\tau] + M_\tau^\perp) \setminus M_\tau^\perp$ such that $\tilde{F}_{(\mathbb{R}, v)}^{M_\tau} = F_{(Y, \bar{v})}^{M_\tau}$. And for every (Y, \bar{v}) with $Y \in L_d^q(\mathcal{F}_T)_+$ and $\bar{v} \in (\mathbb{E} [Y | \mathcal{F}_\tau] + M_\tau^\perp) \setminus M_\tau^\perp$ there exists $(\mathbb{Q}, w_\tau) \in \mathcal{W}_\tau$ such that $F_{(Y, \bar{v})}^{M_\tau} = \tilde{F}_{(\mathbb{Q}, w_\tau)}^{M_\tau}$ by setting $w_\tau = \mathbb{E} [Y | \mathcal{F}_\tau]$ and $\frac{d\mathbb{Q}_i}{d\mathbb{P}} = \frac{Y_i}{\mathbb{E}[Y_i]}$ if $\mathbb{E}[Y_i] > 0$ (and $\frac{d\mathbb{Q}_i}{d\mathbb{P}} \in L^1(\mathcal{F}_T)_+$ with $\mathbb{E} \left[\frac{d\mathbb{Q}_i}{d\mathbb{P}} \right] = 1$ arbitrary if $\mathbb{E}[Y_i] = 0$). Therefore it remains to show that there exists a $w_t \in L_d^q(\mathcal{F}_t)$ such that $w_\tau = \tilde{w}_t^\tau(\mathbb{Q}, w_t)$ and $(\mathbb{Q}, w_t) \in \mathcal{W}_t$. Let $w_t := \mathbb{E} [w_\tau | \mathcal{F}_t] = \mathbb{E} [Y | \mathcal{F}_t]$.

- (a) Show $w_\tau = \tilde{w}_t^\tau(\mathbb{Q}, w_t)$, i.e. show $(w_\tau)_i = \frac{(w_t)_i \mathbb{E} \left[\frac{d\mathbb{Q}_i}{d\mathbb{P}} \middle| \mathcal{F}_\tau \right]}{\mathbb{E} \left[\frac{d\mathbb{Q}_i}{d\mathbb{P}} \middle| \mathcal{F}_t \right]}$ for every $i = 1, \dots, d$. If $\mathbb{E}[Y_i] = 0$ then $(w_t)_i = 0$ and $(w_\tau)_i = 0$, and thus $(w_\tau)_i = (\tilde{w}_t^\tau(\mathbb{Q}, w_t))_i$. If $\mathbb{E}[Y_i] > 0$ then

$$\begin{aligned} \frac{(w_t)_i \mathbb{E} \left[\frac{d\mathbb{Q}_i}{d\mathbb{P}} \middle| \mathcal{F}_\tau \right]}{\mathbb{E} \left[\frac{d\mathbb{Q}_i}{d\mathbb{P}} \middle| \mathcal{F}_t \right]} &= \frac{\mathbb{E} [Y_i | \mathcal{F}_t] \mathbb{E} [Y_i | \mathcal{F}_\tau] / \mathbb{E} [Y_i]}{\mathbb{E} [Y_i | \mathcal{F}_t] / \mathbb{E} [Y_i]} \\ &= \begin{cases} \mathbb{E} [Y_i | \mathcal{F}_\tau] (\omega) & \text{if } \mathbb{E} [Y_i | \mathcal{F}_t] (\omega) > 0 \\ 0 = \mathbb{E} [Y_i | \mathcal{F}_\tau] (\omega) & \text{if } \mathbb{E} [Y_i | \mathcal{F}_t] (\omega) = 0 \end{cases} = (w_\tau)_i. \end{aligned}$$

- (b) Show $(\mathbb{Q}, w_t) \in \mathcal{W}_t$

- i. Show $w_t \in ((M_t)_+)^+ \setminus M_t^\perp$.

A. Let $m_t \in (M_t)_+$, then $\mathbb{E} [w_t^\top m_t] = \mathbb{E} \left[\mathbb{E} [w_\tau | \mathcal{F}_t]^\top m_t \right] = \mathbb{E} [w_\tau^\top m_t] \geq 0$ by the tower property, $(M_t)_+ \subseteq (M_\tau)_+$ and $w_\tau \in ((M_\tau)_+)^+$.

B. Since $(\mathbb{Q}, w_\tau) \in \mathcal{W}_\tau$, in particular since $w_\tau \notin M_\tau^\perp$ there exists $m_\tau \in M_\tau$ such that $\mathbb{E} [w_\tau^\top m_\tau] \neq 0$. Then $\mathbb{E}^\mathbb{Q} [m_\tau | \mathcal{F}_t] \in M_t$ by $M_t \supseteq M_\tau \cap L_d^p(\mathcal{F}_t)$ and $\tilde{M}_\tau[\omega_1] = \tilde{M}_\tau[\omega_2]$ for almost every $\omega_1, \omega_2 \in \Omega$. Therefore, $\mathbb{E} [w_t^\top \mathbb{E}^\mathbb{Q} [m_\tau | \mathcal{F}_t]] = \mathbb{E} [\tilde{w}_t^\tau(\mathbb{Q}, w_t)^\top m_\tau] = \mathbb{E} [w_\tau^\top m_\tau] \neq 0$.

$$\text{ii. } \tilde{w}_t^T(\mathbb{Q}, w_t) = \tilde{w}_\tau^T(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w_t)) = \tilde{w}_\tau^T(\mathbb{Q}, w_\tau) \in L_d^q(\mathcal{F}_T)_+.$$

□

The following corollary of lemma 8.1 uses the above result applied to penalty functions instead of the functionals $\tilde{F}_{(\cdot, \cdot)}[\cdot]$.

Corollary 8.2. *Let assumption 3.1 be satisfied. Then, for any $(\mathbb{R}, v) \in \mathcal{W}_\tau$ there exists $(\mathbb{Q}, w) \in \mathcal{W}_t$ such that*

$$-\alpha_\tau^{\min}(\mathbb{R}, v) = -\alpha_\tau^{\min}(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w))$$

for any times $t, \tau \in \{0, 1, \dots, T\}$ with $t < \tau$.

Proof.

$$\begin{aligned} -\alpha_\tau^{\min}(\mathbb{R}, v) &= \text{cl} \bigcup_{Z \in A_\tau} \tilde{F}_{(\mathbb{R}, v)}^{M_\tau}[Z] \\ &= \text{cl} \bigcup_{Z \in A_\tau} \tilde{F}_{(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w))}^{M_\tau}[Z] \\ &= -\alpha_\tau^{\min}(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)), \end{aligned} \tag{8.1}$$

where equation (8.1) is a result of lemma 8.1. □

Lemma 8.1 and corollary 8.2 show that for a given penalty function $-\alpha_\tau^{\min}$ the set of dual variables $\{(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) : (\mathbb{Q}, w) \in \mathcal{W}_t\}$, $t \leq \tau$ defines the same closed and convex risk measure at time τ as the set of dual variables \mathcal{W}_τ , that is

$$\begin{aligned} R_\tau(X) &= \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_t} \left[-\alpha_\tau^{\min}(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) + \left(\mathbb{E}^\mathbb{Q}[-X | \mathcal{F}_\tau] + G_\tau(\tilde{w}_t^\tau(\mathbb{Q}, w)) \right) \right] \cap M_\tau \\ &= \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_\tau} \left[-\alpha_\tau^{\min}(\mathbb{Q}, w) + \left(\mathbb{E}^\mathbb{Q}[-X | \mathcal{F}_\tau] + G_\tau(w) \right) \right] \cap M_\tau. \end{aligned}$$

The following lemma, about the expectation of minimal penalty functions, is an extension of lemma 2.6 in [17]. As a set-valued operation, this theorem gives a set-valued version of when the conditional expectation of an infimum is equivalent to the infimum of the conditional expectation. The proof of the lemma is a simplified version of the proof of lemma 2.6 in [17] since the sets $\{\mathbb{E}^\mathbb{Q}[X | \mathcal{F}_t] + G_t(w)\}$ are shifted half spaces for any $X \in A_t$ and a fixed $(\mathbb{Q}, w) \in \mathcal{W}_t$ and thus are completely ordered, in contrast to the scalar case, where the points $\mathbb{E}^\mathbb{Q}[X | \mathcal{F}_t]$ under consideration are not completely ordered.

Lemma 8.3. *For any times $t, \tau \in \{0, 1, \dots, T\}$ with $t \leq \tau$, and if R_t is a closed convex risk measure under assumption 3.1, then for any $(\mathbb{Q}, w) \in \mathcal{W}_t$, it follows that*

$$\mathbb{E}^\mathbb{Q}[-\alpha_\tau^{\min}(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) | \mathcal{F}_t] = \text{cl} \bigcup_{X \in A_\tau} \left(\mathbb{E}^\mathbb{Q}[X | \mathcal{F}_t] + G_t(w) \right) \cap M_t.$$

Proof. Let $(\mathbb{Q}, w) \in \mathcal{W}_t$. Then, by lemma 8.1, $(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) \in \mathcal{W}_\tau$. It holds

$$\begin{aligned} -\alpha_\tau^{\min}(\mathbb{Q}, \tilde{w}_t^\tau(\mathbb{Q}, w)) &= \text{cl} \bigcup_{X \in A_\tau} \left(\mathbb{E}^\mathbb{Q}[X | \mathcal{F}_\tau] + G_\tau(\tilde{w}_t^\tau(\mathbb{Q}, w)) \right) \cap M_\tau \\ &= \text{cl} \bigcup_{X \in A_\tau} \left\{ u \in M_\tau : \mathbb{E} \left[\tilde{w}_t^\tau(\mathbb{Q}, w)^\top \mathbb{E}^\mathbb{Q}[X | \mathcal{F}_\tau] \right] \leq \mathbb{E} \left[\tilde{w}_t^\tau(\mathbb{Q}, w)^\top u \right] \right\} \\ &= \text{cl} \bigcup_{X \in A_\tau} \left\{ u \in M_\tau : \mathbb{E} \left[w^\top \mathbb{E}^\mathbb{Q}[X | \mathcal{F}_t] \right] \leq \mathbb{E} \left[w^\top \mathbb{E}^\mathbb{Q}[u | \mathcal{F}_t] \right] \right\} \\ &= \left\{ u \in M_\tau : \inf_{X \in A_\tau} \mathbb{E} \left[w^\top \mathbb{E}^\mathbb{Q}[X | \mathcal{F}_t] \right] \leq \mathbb{E} \left[w^\top \mathbb{E}^\mathbb{Q}[u | \mathcal{F}_t] \right] \right\}. \end{aligned}$$

Taking the conditional expectation on both sides yields

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}} [-\alpha_{\tau}^{\min}(\mathbb{Q}, \tilde{w}_t^{\tau}(\mathbb{Q}, w)) | \mathcal{F}_t] &= \left\{ \mathbb{E}^{\mathbb{Q}} [u | \mathcal{F}_t] : u \in M_{\tau}, \inf_{X \in A_{\tau}} \mathbb{E} \left[w^{\top} \mathbb{E}^{\mathbb{Q}} [X | \mathcal{F}_t] \right] \leq \mathbb{E} \left[w^{\top} \mathbb{E}^{\mathbb{Q}} [u | \mathcal{F}_t] \right] \right\} \\
&= \left\{ u \in M_t : \inf_{X \in A_{\tau}} \mathbb{E} \left[w^{\top} \mathbb{E}^{\mathbb{Q}} [X | \mathcal{F}_t] \right] \leq \mathbb{E} \left[w^{\top} u \right] \right\} \\
&= \text{cl} \bigcup_{X \in A_{\tau}} \left(\mathbb{E}^{\mathbb{Q}} [X | \mathcal{F}_t] + G_t(w) \right) \cap M_t.
\end{aligned}$$

□

We conclude our discussion on how dual variables across time are related by considering the conditional expectation of random halfspaces. In particular, we demonstrate that the \mathbb{Q} -conditional expectation (at time t) of the halfspace defined by $\tilde{w}_t^{\tau}(\mathbb{Q}, w)$ is given by the halfspace defined by w .

Proposition 8.4. *Let $t, \tau \in \{0, 1, \dots, T\}$ with $t < \tau$ and let $\mathbb{Q} \in \mathcal{M}_d(\mathbb{P})$ and $w \in L_d^q(\mathcal{F}_t)$. Then,*

$$\mathbb{E}^{\mathbb{Q}} [G_{\tau}(\tilde{w}_t^{\tau}(\mathbb{Q}, w)) | \mathcal{F}_t] = G_t(w).$$

Proof.

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}} [G_{\tau}(\tilde{w}_t^{\tau}(\mathbb{Q}, w)) | \mathcal{F}_t] &= \left\{ \mathbb{E}^{\mathbb{Q}} [u | \mathcal{F}_t] : u \in L_d^p(\mathcal{F}_{\tau}), 0 \leq \mathbb{E} \left[\tilde{w}_t^{\tau}(\mathbb{Q}, w)^{\top} u \right] \right\} \\
&= \left\{ \mathbb{E}^{\mathbb{Q}} [u | \mathcal{F}_t] : u \in L_d^p(\mathcal{F}_{\tau}), 0 \leq \mathbb{E} \left[\mathbb{E} \left[\tilde{w}_t^{\tau}(\mathbb{Q}, w)^{\top} u \mid \mathcal{F}_t \right] \right] \right\} \\
&= \left\{ \mathbb{E}^{\mathbb{Q}} [u | \mathcal{F}_t] : u \in L_d^p(\mathcal{F}_{\tau}), \right. \\
&\quad \left. 0 \leq \mathbb{E} \left[\mathbb{E} \left[\left(\text{diag}(w) \text{diag} \left(\mathbb{E} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{F}_t \right] \right)^{-1} \frac{d\mathbb{Q}}{d\mathbb{P}} \right)^{\top} u \mid \mathcal{F}_t \right] \right] \right\} \\
&= \left\{ \mathbb{E}^{\mathbb{Q}} [u | \mathcal{F}_t] : u \in L_d^p(\mathcal{F}_{\tau}), 0 \leq \mathbb{E} \left[w^{\top} \mathbb{E}^{\mathbb{Q}} [u | \mathcal{F}_t] \right] \right\} \\
&= \left\{ u \in L_d^p(\mathcal{F}_t) : 0 \leq \mathbb{E} \left[w^{\top} u \right] \right\} \\
&= G_t(w).
\end{aligned}$$

□

8.2 On the sum of closed acceptance sets

When considering multi-portfolio time consistency for closed risk measures we need to guarantee that the composed risk measures are closed, or else the recursive form would fail to hold. In particular, this would be true if the sum of acceptance sets are themselves closed. Note that an application of the Dieudonne theorem, characterizing when the sum of two sets is closed, is not possible here as it requires strong assumptions that are not satisfied in our case. However, since the sets are acceptance set, closedness of the summand is all one needs to assume. In the scalar framework this condition is not considered since the value of the risk measure is an (essential) infimum and thus the closure of the acceptance set gives the same risk compensating value as the original acceptance set.

Lemma 8.5. *Let $(A_t)_{t=0}^T$ be a sequence of closed acceptance sets. Let M_{t+1} be the set of eligible portfolios at time $t+1$ (a closed linear subspace of $L_d^p(\mathcal{F}_t)$). Then, $A_{t,t+1}^{M_{t+1}} + A_{t+1}$ is closed.*

Proof. Consider a net $(X_i)_{i \in I}$ (or a sequence if $p < \infty$) with index set I satisfying $X_i \in A_{t,t+1}^{M_{t+1}} + A_{t+1}$ and $\lim_i X_i = X \in L_d^p(\mathcal{F}_T)$. We need to show that $X \in A_{t,t+1}^{M_{t+1}} + A_{t+1}$. Since A_{t+1} is closed, R_{t+1} is closed (proposition 2.11 (vii) in [16]) and thus (see proposition 2.34 in [31]) it holds

$$R_{t+1}(X) \supseteq \bigcap_{i \in I} \text{cl} \bigcup_{j \geq i} R_{t+1}(X_j) = \liminf_{i \in I} R_{t+1}(X_i).$$

Lemma 3.6 (i) in [16], using that $X_i \in A_{t,t+1}^{M_{t+1}} + A_{t+1}$ for all $i \in I$, yields

$$\begin{aligned} 0 &\in \bigcap_{i \in I} \bigcup_{Y \in R_{t+1}(X_i)} R_t(-Y) \\ &\subseteq \bigcap_{i \in I} \bigcup_{Y \in \text{cl} \bigcup_{j \geq i} R_{t+1}(X_j)} R_t(-Y) \\ &= \bigcup_{Y \in \bigcap_{i \in I} \text{cl} \bigcup_{j \geq i} R_{t+1}(X_j)} R_t(-Y) \\ &\subseteq \bigcup_{Y \in R_{t+1}(X)} R_t(-Y). \end{aligned} \tag{8.2}$$

This implies $X \in A_{t,t+1}^{M_{t+1}} + A_{t+1}$ by lemma 3.6 (i) in [16]. Equation (8.2) follows from:

1. If $u \in \bigcup_{Y \in \bigcap_{i \in I} \text{cl} \bigcup_{j \geq i} R_{t+1}(X_j)} R_t(-Y)$ then trivially $u \in \bigcap_{i \in I} \bigcup_{Y \in \text{cl} \bigcup_{j \geq i} R_{t+1}(X_j)} R_t(-Y)$.
2. If $u \in \bigcap_{i \in I} \bigcup_{Y \in \text{cl} \bigcup_{j \geq i} R_{t+1}(X_j)} R_t(-Y)$ then for every $i \in I$ there exists a net $Y_{i,\alpha} \in \bigcup_{j \geq i} R_{t+1}(X_j)$ converging to $Y_i \in \text{cl} \bigcup_{j \geq i} R_{t+1}(X_j)$ such that $u \in R_t(-Y_i)$. Trivially $\bigcup_{j \geq m} R_{t+1}(X_j) \subseteq \bigcup_{j \geq i} R_{t+1}(X_j)$ for every $i, m \in I$ such that $i \leq m$, so we can let $Y_i = Y_m$ for every $i \leq m$. Therefore, there exists a $Y \in M_{t+1}$ such that for every $i \in I$ we have $Y \in \bigcup_{j \geq i} R_{t+1}(X_j)$ such that $u \in R_t(-Y)$, i.e. $u \in \bigcup_{Y \in \bigcap_{i \in I} \text{cl} \bigcup_{j \geq i} R_{t+1}(X_j)} R_t(-Y)$.

□

Moreover, when applying lemma 3.2 to the proof of theorem 3.3 we need not only the sum of closed convex acceptance sets to be closed, but also to be a closed convex acceptance set itself. This is given in the following lemma.

Lemma 8.6. *Let $(A_t)_{t=0}^T$ be a sequence of closed convex normalized acceptance sets, and let $M_t \subseteq M_{t+1}$ for all times $t = 0, 1, \dots, T-1$. Assume $A_{t,t+1}^{M_{t+1}} + A_{t+1} \subseteq A_t$, then $A_{t,t+1}^{M_{t+1}} + A_{t+1}$ is a closed convex acceptance set at time t .*

Proof. $A_{t,t+1}^{M_{t+1}} + A_{t+1}$ is closed by lemma 8.5 and convex since both $A_{t,t+1}^{M_{t+1}}$ and A_{t+1} are convex. Let us check the properties of acceptance sets (see definition 2.2).

1. $A_{t,t+1}^{M_{t+1}} + A_{t+1} \subseteq L_d^p(\mathcal{F}_T)$ trivially.
2. $M_t \cap (A_{t,t+1}^{M_{t+1}} + A_{t+1}) \supseteq M_t \cap M_{t+1} \cap A_t \neq \emptyset$ since $0 \in A_{t+1}$ (by closed and normalized), $M_t \cap A_t \neq \emptyset$, and $M_t \cap M_{t+1} = M_t$.
3. $M_t \cap (L_d^p(\mathcal{F}_T) \setminus \{A_{t,t+1}^{M_{t+1}} + A_{t+1}\}) \supseteq M_t \cap (L_d^p(\mathcal{F}_T) \setminus A_t) \neq \emptyset$ by $A_{t,t+1}^{M_{t+1}} + A_{t+1} \subseteq A_t$.
4. $A_{t,t+1}^{M_{t+1}} + A_{t+1} + L_d^p(\mathcal{F}_T)_+ \subseteq A_{t,t+1}^{M_{t+1}} + A_{t+1}$ trivially.

□

8.3 Stepped risk measures

In this section, we consider the dual representation of closed convex and coherent stepped risk measures $R_{t,\tau} : M_\tau \rightarrow \mathcal{P}(M_t; (M_t)_+)$. This is used in sections 3 and 4 as the stepped penalty functions and stepped sets of dual variables play a role when discussing equivalent characterizations of multi-portfolio time consistency. For the dual representation we will use set-valued duality defined in [22] analogously as for conditional risk measures in section 4 of [16].

Given a risk measure $R_t : L_d^p(\mathcal{F}_T) \rightarrow \mathcal{P}(M_t; (M_t)_+)$, a stepped risk measure $R_{t,\tau}(X) := \{u \in M_t : X + u \in A_{t,\tau}^{M_\tau}\}$ for $X \in M_\tau$ is the restriction of R_t to M_τ , i.e. $R_{t,\tau} = R_t|_{M_\tau}$. Therefore, if R_t is closed convex (coherent) then $R_{t,\tau}$ is closed convex (coherent). Furthermore, if R_t is $L_d^p(\mathcal{F}_T)_+$ -monotone, then $R_{t,\tau}$ is $(M_\tau)_+$ -monotone.

Lemma 8.7. *Let R_t be a closed convex risk measure. The set of dual variables for $R_{t,\tau} : M_\tau \rightarrow \mathcal{P}(M_t; (M_t)_+)$ with $t < \tau$ under assumption 3.1 is given by*

$$\mathcal{W}_{t,\tau} = \left\{ (\mathbb{Q}, w) \in \mathcal{M}_d(\mathbb{P}) \times \left(((M_t)_+)^+ \setminus M_t^\perp \right) : \tilde{w}_t^\tau(\mathbb{Q}, w) \in ((M_\tau)_+)^+ \right\}.$$

Proof. By $(M_\tau)_+$ -monotonicity and the logic of proposition 4.4 in [16] the (classical) stepped dual variables are given by $\{(Y, v) : Y \in ((M_\tau)_+)^+, v \in (\mathbb{E}[Y | \mathcal{F}_t] + M_t^\perp) \setminus M_t^\perp\}$. Then it remains to show that for any dual pair (Y, v) there exists a $(\mathbb{Q}, w) \in \mathcal{W}_{t,\tau}$ such that $F_{(Y,v)}^{M_t}[X] = \tilde{F}_{(\mathbb{Q},w)}^{M_t}[X]$ for any $X \in M_\tau$, and vice versa.

1. Let $(\mathbb{Q}, w) \in \mathcal{W}_{t,\tau}$. Then, we will show that there exists a dual pair

$$(Y, v) \in \left\{ (Y, v) : Y \in ((M_\tau)_+)^+, v \in (\mathbb{E}[Y | \mathcal{F}_t] + M_t^\perp) \setminus M_t^\perp \right\}$$

such that $F_{(Y,v)}^{M_t}[X] = \tilde{F}_{(\mathbb{Q},w)}^{M_t}[X]$ for any $X \in M_\tau$. Let $Y = \tilde{w}_t^\tau(\mathbb{Q}, w) \in ((M_\tau)_+)^+$ (by remark 8.8 and lemma 8.1 (i)), therefore $\mathbb{E}[X^\top Y] = \mathbb{E}[\tilde{w}_t^\tau(\mathbb{Q}, w)^\top X] = \mathbb{E}[w^\top \mathbb{E}^\mathbb{Q}[X | \mathcal{F}_t]]$ and $\mathbb{E}[Y | \mathcal{F}_t] = w$. From $w \in ((M_t)_+)^+ \setminus M_t^\perp$ we can rewrite $w = w_{((M_t)_+)^+} + w_{M_t^\perp}$. Thus $v = w_{((M_t)_+)^+} = w - w_{M_t^\perp} \in \mathbb{E}[Y | \mathcal{F}_t] + M_t^\perp$. Finally, $w \notin M_t^\perp$ implies $v \notin M_t^\perp$, and $\mathbb{E}[w^\top u] = \mathbb{E}[v^\top u]$ for every $u \in M_t$ since $w \in v + M_t^\perp$.

2. Let $(Y, v) \in \{(Y, v) : Y \in ((M_\tau)_+)^+, v \in (\mathbb{E}[Y | \mathcal{F}_t] + M_t^\perp) \setminus M_t^\perp\}$. We want to show there exists a $(\mathbb{Q}, w) \in \mathcal{W}_{t,\tau}$ such that $F_{(Y,v)}^{M_t}[X] = \tilde{F}_{(\mathbb{Q},w)}^{M_t}[X]$ for any $X \in M_\tau$. Let $w \in \mathbb{E}[(Y + M_\tau^\perp) \cap L_d^q(\mathcal{F}_\tau)_+ | \mathcal{F}_t]$ (which is nonempty), i.e., $w = \mathbb{E}[Y + m^\perp | \mathcal{F}_t]$ for some $m^\perp \in M_\tau^\perp$ and $Y + m^\perp \in L_d^q(\mathcal{F}_\tau)_+$. Then it can easily be seen that $w \in v + M_t^\perp$ for $v \in (\mathbb{E}[Y | \mathcal{F}_t] + M_t^\perp) \setminus M_t^\perp \subseteq ((M_t)_+)^+$. Thus $w \in ((M_t)_+)^+ + M_t^\perp$ and with $v \notin M_t^\perp$ this implies $w \in ((M_t)_+)^+ \setminus M_t^\perp$. From $w \in v + M_t^\perp$ it follows that $\mathbb{E}[w^\top u] = \mathbb{E}[v^\top u]$ for every $u \in M_t$.

Additionally, choose $\mathbb{Q} \in \mathcal{M}_d(\mathbb{P})$ such that $\mathbb{E}\left[\frac{d\mathbb{Q}_i}{d\mathbb{P}} \middle| \mathcal{F}_\tau\right] = \frac{Y_i + m_i^\perp}{\mathbb{E}[Y_i + m_i^\perp]}$ if $\mathbb{E}[Y_i + m_i^\perp] > 0$ and arbitrarily in $L_d^1(\mathcal{F}_\tau)_+$ such that $\mathbb{E}\left[\frac{d\mathbb{Q}_i}{d\mathbb{P}}\right] = 1$ if $\mathbb{E}[Y_i + m_i^\perp] = 0$. Then $\tilde{w}_t^\tau(\mathbb{Q}, w) = Y + m^\perp \in ((M_\tau)_+)^+ + M_\tau^\perp \subseteq ((M_\tau)_+)^+$. And thus it follows that $\mathbb{E}[w^\top \mathbb{E}^\mathbb{Q}[X | \mathcal{F}_t]] = \mathbb{E}[\tilde{w}_t^\tau(\mathbb{Q}, w)^\top X] = \mathbb{E}[Y^\top X]$ for every $X \in M_\tau$.

□

Remark 8.8. For any choice of eligible portfolios M_t satisfying assumption 3.1 it trivially follows that $\mathcal{W}_{t,\tau} \supseteq \mathcal{W}_t$ for any $t < \tau$.

Remark 8.9. If we denote $\mathcal{W}_{t,\tau}[M, N]$ to be the set of stepped dual variables with eligible spaces M at time t and N at time τ , and $\mathcal{W}_t[M]$ denotes the set of dual variables with respect to the eligible space M at time t . Then, an inspection of the proof of lemma 4.5 from [16] shows that, for any choice of eligible portfolios M_t , $\mathcal{W}_{t,\tau}[M_t, L_d^p(\mathcal{F}_\tau)] = \mathcal{W}_t[M_t]$ holds true.

The lemma below gives a dual representation for closed convex stepped risk measures. In particular, it demonstrates that the minimal stepped penalty function as defined in (3.1) can be used in a dual representation to define a closed convex stepped risk measures.

Lemma 8.10. *The dual representation for any $(M_\tau)_+$ -monotone closed convex stepped risk measure $R_{t,\tau} : M_\tau \rightarrow \mathcal{P}(M_t; (M_t)_+)$ with $t < \tau$ is given by*

$$R_{t,\tau}(X) = \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_{t,\tau}} \left[-\alpha_{t,\tau}^{\min}(\mathbb{Q}, w) + \left(\mathbb{E}^{\mathbb{Q}}[-X | \mathcal{F}_t] + G_t(w) \right) \cap M_t \right]$$

for any $X \in M_\tau$ where

$$-\alpha_{t,\tau}^{\min}(\mathbb{Q}, w) = \text{cl} \bigcup_{X \in A_{t,\tau}^{M_\tau}} \left(\mathbb{E}^{\mathbb{Q}}[X | \mathcal{F}_t] + G_t(w) \right) \cap M_t.$$

Proof. This is an adaption of theorem 2.4 to stepped risk measures using lemma 8.7. □

Finally, we will use the above results to give a dual representation for closed coherent stepped risk measures.

Corollary 8.11. *The dual representation for any $(M_\tau)_+$ -monotone closed coherent stepped risk measure $R_{t,\tau} : M_\tau \rightarrow \mathcal{P}(M_t; (M_t)_+)$ with $t < \tau$ is given by*

$$R_{t,\tau}(X) = \bigcap_{(\mathbb{Q}, w) \in \mathcal{W}_{t,\tau}^{\max}} \left(\mathbb{E}^{\mathbb{Q}}[-X | \mathcal{F}_t] + G_t(w) \right) \cap M_t$$

for any $X \in M_\tau$ where

$$\mathcal{W}_{t,\tau}^{\max} = \left\{ (\mathbb{Q}, w) \in \mathcal{W}_{t,\tau} : \tilde{w}_t^\tau(\mathbb{Q}, w) \in (A_{t,\tau}^{M_\tau})^+ \right\}.$$

Proof. Note that $-\alpha_{t,\tau}^{\min}(\mathbb{Q}, w) = \text{cl} \bigcup_{X \in A_{t,\tau}^{M_\tau}} \left(\mathbb{E}^{\mathbb{Q}}[X | \mathcal{F}_t] + G_t(w) \right) \cap M_t = G_t(w) \cap M_t$ if and only if for every $X \in A_{t,\tau}^{M_\tau}$ we have

$$\mathbb{E} \left[w^\top \mathbb{E}^{\mathbb{Q}}[X | \mathcal{F}_t] \right] = \mathbb{E} \left[\tilde{w}_t^\tau(\mathbb{Q}, w)^\top X \right] \geq 0,$$

i.e. $\tilde{w}_t^\tau(\mathbb{Q}, w) \in (A_{t,\tau}^{M_\tau})^+$. Thus, for a $(M_\tau)_+$ -monotone closed coherent stepped risk measure $R_{t,\tau}$ with $t, \tau \in \{0, 1, \dots, T\}$ and $t < \tau$ it holds that for any $(\mathbb{Q}, w) \in \mathcal{W}_{t,\tau}$

$$-\alpha_{t,\tau}^{\min}(\mathbb{Q}, w) = G_t(w) \cap M_t \Leftrightarrow \tilde{w}_t^\tau(\mathbb{Q}, w) \in (A_{t,\tau}^{M_\tau})^+.$$

An application of lemma 8.7 provides the desired result. □

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